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Two-Stage Budgeting through Utility Maximization and Expenditure Minimization under Homogeneous Separability

Ashley Atoo, Giulio Codognato

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Ashley Atoo*
Giulio Codognato†

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Abstract

In this paper, we formulate a hybrid version of the two-stage procedure for the utility maximization problem of a consumer proposed by Green (1964). In particular, we replace the second stage of his procedure with a problem of expenditure minimization. Our procedure allows us to specify the price indices of each group of commodities as the minimum expenditure to achieve an utility level equal to one at the prices of the commodities belonging to that group.
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1 Introduction

Green (1964) presented in a systematic way some seminal contributions to the theory of aggregation in economic analysis (we refer directly to his book for a survey of this path-breaking literature).

In particular, he analyzed the conditions under which the utility maximization problem of a consumer can be split into two stages, by grouping commodities and determining a quantity index and a price index for each

*Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy.

†Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy, and Economix, UPL, Univ Paris Nanterre, CNRS, F92000 Nanterre, France.

group of commodities. One of these conditions, called homogeneous separability, requires that the utility function of the consumer can be expressed as a function of quantity indices, each representing a group of commodities as a function homogeneous of degree one defined on the commodities in that group. In order to define a two-stage procedure, a price index must be associated to each group of commodities also expressed as a function of the prices of the commodities in that group. Given the consumer's income, Green (1964) considered a two-stage utility maximization procedure whose first stage determines, through the solution of a maximization problem, the quantity indices of each group of commodities, given the corresponding price indices and the consumer's income, and whose second stage determines, through the solution of a maximization problem, the quantities of the commodities belonging to each group given the prices of the commodities in that group and the product between the group quantity and price indices as a budget constraint. Under some standard regularity assumptions which guarantee the uniqueness of the solutions to the maximization problems, Green (1964) defined as consistent that two-stage procedure whose unique solution coincides with the solution of the consumer's utility maximization problem.

One of the most successful applications of the two-stage maximization procedure systematized by Green (1964) was provided by Dixit and Stiglitz (1977) in a seminal article which introduced a new approach to the theory of monopolistic competition, generating an impressive stream of literature. In particular, they split commodities into two groups, one containing just a *numéraire* commodity and the other containing all other commodities. Then, the whole demand properties of their model of monopolistic competition were derived through the Green two-stage maximization procedure sketched above, under the crucial assumption of homogeneity of degree one of the quantity indices.

Lloyd (1977) considered the dual of the Green two-stage maximization procedure: a two-stage minimization procedure. He fully developed the theoretical background of this procedure and its fruitful applications.

d'Aspremont and Dos Santos Ferreira (2016) reconsidered the model of monopolistic competition introduced by Dixit and Stiglitz (1977) and replaced their procedure with a hybrid one consisting of a first stage where the consumer minimizes the expenditure to achieve a certain level of the quantity index associated to the non-*numéraire* commodities and a second stage which determines, through the solution of a maximization problem, the quantities of each group of commodities. Nevertheless, they left unspecified the properties of the quantity and price indices associated with their hybrid two-stage maximization procedure.

In this paper, we also formulate a hybrid version of the Green two-stage procedure. In particular, we solve his utility maximization problem maintaining all his assumptions and using a two-stage procedure whose first stage determines, through the solution of a maximization problem, the quantity indices of each group of commodities, given the corresponding price indices and the consumer's income, and whose second stage determines, through the solution of an expenditure minimization problem, the quantities of the commodities belonging to each group, given the prices of the commodities in that group, which minimize the expenditure to achieve the level of the corresponding quantity index, determined as a solution of the first stage maximization problem. We show the consistency of this procedure, under the assumption, neglected by Green (1964) and Lloyd (1977), that the solutions to all the optimization problems are interior. The advantage of the two-stage procedure we propose is that, thanks to homogeneous separability, in the first stage, the quantity indices are nonnegative and the price indices of each group of commodities correspond to the minimum expenditure to achieve an utility level equal to one at the prices of the commodities belonging to that group.

Finally, we note that, in the Dixit and Stiglitz framework, under the assumption of homogeneous separability, our two-stage maximization procedure is the dual of that proposed by d'Aspremont and Dos Santos Ferreira (2016).

2 Mathematical model

We consider a consumer who consumes n commodities. Let the vector $x = (x_1, \ldots, x_n) \in R^n_+$ denote a bundle of commodities. The preferences of the consumer are represented by a utility function $u(x)$. We make the following assumption on the function u.

Assumption 1. *The function u is continuous and twice continuously differentiable.*

We now provide a definition of weak separability of the function u (see Lloyd (1977)).

Definition 1. *The function u is said to be weakly separable with respect to the partition of the n commodities into m disjoint and exhaustive subsets*

$N_1, \dots, N_m$ *if*

$$u(x) = V(v^1(\bar{x}_1), \dots, v^j(\bar{x}_j), \dots, v^m(\bar{x}_m),$$

where the functions v^j *are continuous and twice continuously differentiable,* $\bar{x}_j = (x_{j1}, \dots, x_{jn_j})$, $j = 1, \dots, m < n$, *and* $\sum_{j=1}^{m} n_j = n$.

The notion of weak separability can be strengthened imposing the requirement that the functions v^j are homogeneous of degree one for each $j = 1, \dots, m$ (see Lloyd (1977)).

Definition 2. *The function* u *is said to be homogeneously separable with respect to the partition of the* n *commodities into* m *disjoint and exhaustive subsets* $N_1, \dots, N_m$ *if it is weakly separable and the functions* v^j *are homogeneous of degree one, for each* $j = 1, \dots, m$.

We make the following assumption on the function u.

Assumption 2. *The function* u *is homogeneously separable.*

Henceforth, given Assumption 2, we shall rename, with some abuse of notation, the arguments of the utility function u, according to their partition, as follows

$$x = (x_{11}, \dots, x_{1n_1}, \dots, x_{j1}, \dots, x_{jn_j}, \dots, x_{m1}, \dots, x_{mn_m}).$$

Let $u_{ji} = \frac{\partial u}{\partial x_{ji}}$, $j = 1, \dots, m$, $i = 1, \dots, n_j$ and $u_{ji,rk} = \frac{\partial^2 u}{\partial x_{ji} \partial x_{rk}}$, $j = 1, \dots, m$, $i = 1, \dots, n_j$, $r = 1, \dots, m$, $k = 1, \dots, n_k$.

We can now introduce further regularity conditions on the function u (see Green (1964)).

Assumption 3. $u_{ji} > 0$, $j = 1, \dots, m$, $i = 1, \dots, n_j$ *and the principal minors of order* q $(q \geq 3)$ *of the matrix*

$$A = \begin{bmatrix} 0 & u_{11} & \dots & u_{ji} & \dots & u_{mn_m} \\ u_{11} & u_{11,11} & \dots & u_{11,ji} & \dots & u_{11,mn_m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ u_{ji} & u_{ji,11} & \dots & u_{ji,ji} & \dots & u_{ji,mn_m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ u_{mn_m} & u_{mn_m,11} & \dots & u_{mn_m,ji} & \dots & u_{mn_m,mn_m} \end{bmatrix}$$

have the sign of $(-1)^{q+1}$.

According to Assumption 3, the function u is strongly increasing and strictly quasi-concave.

Let $p = (p_{11}, \ldots, p_{1n_1}, \ldots, p_{j1}, \ldots, p_{jn_j}, \ldots, p_{m1}, \ldots, p_{mn_m}) \in R^n_{++}$ be a vector of prices and let I be the income of the consumer. Moreover let $\bar{p}_j = (p_{j1}, \ldots, p_{jn_j})$, $j = 1, \ldots, m$.

Consider the following utility maximization problem

$$\begin{aligned} &\max_x u(x) \\ &\text{subject to} \\ &px = I. \end{aligned} \tag{1}$$

There exists a unique solution to the utility maximization problem (1) $\hat{\bar{x}}_j$, $j = 1, \ldots, m$, as the utility function is continuous, strongly increasing, and strictly quasi-concave (see Propositions 3.D.1 and 3.D.2 in Mas-Colell et al. (1995)).

Let $V_j = \frac{\partial V}{\partial v^j}$, $j = 1, \ldots, m$, and $V_{jr} = \frac{\partial^2 V}{\partial v^j \partial v^r}$, $j = 1, \ldots, m$, $r = 1, \ldots, m$. Moreover, let $v^j_{ji} = \frac{\partial v^j}{\partial x_{ji}}$, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$, and $v^j_{ji,jk} = \frac{\partial^2 v^j}{\partial x_{ji} \partial x_{jk}}$, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$, $k = 1, \ldots, n_j$.

We make the following further assumption on the functions v^j, $j = 1, \ldots, m$.

Assumption 4. $v^j_{ji} > 0$, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$.

The following proposition establishes the regularity conditions of the function V.

Proposition 1. *Under Assumptions 1, 2, 3, and 4, the function V is such that $V_j > 0$, $j = 1, \ldots, m$, and the principal minors of order q ($q \geq 3$) of the matrix*

$$B = \begin{bmatrix} 0 & V_1 & \ldots & V_j & \ldots & V_m \\ V_1 & V_{11} & \ldots & V_{1j} & \ldots & V_{1m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ V_j & V_{j1} & \ldots & V_{jj} & \ldots & V_{jmn} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ V_m & V_{m1} & \ldots & V_{mj} & \ldots & V_{mm} \end{bmatrix}$$

have the sign of $(-1)^{q+1}$.

Proof. We have that $u_{ji} = V_j v^j_{ji}$, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$. Then, it must be that $V_j > 0$, $j = 1, \ldots, m$, as $u_{ji} > 0$, by Assumption 3, and $v^j_{ji} > 0$, by Assumption 4, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$. The principal minors of order q ($q \geq 3$) of the matrix B have the sign of $(-1)^{q+1}$ as the principal minors of order q ($q \geq 3$) of the matrix A have the sign of $(-1)^{q+1}$, by Assumption 3, and the functions v^j are homogeneous of degree one, for each $j = 1, \ldots, m$, by Assumption 2, by Theorem 5 in Green (1964). ■

According to Proposition 1, the function V is strongly increasing and strictly quasi-concave. The following proposition completes the regularity conditions of the functions v^j, $j = 1, \ldots, m$.

Proposition 2. *Under Assumptions 1, 2, 3, and 4, the principal minors of order q ($q \geq 3$) of the matrices*

$$C^j = \begin{bmatrix} 0 & v^j_{j1} & \cdots & v^j_{jk} & \cdots & v^j_{jn_j} \\ v^j_{j1} & v^j_{j1,j1} & \cdots & v^j_{j1,jk} & \cdots & v^j_{j1,jn_j} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ v^j_{jk} & v^j_{ji,j1} & \cdots & v^j_{ji,jk} & \cdots & v^j_{ji,jn_j} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ v^j_{jn_j} & v^j_{jn_j,j1} & \cdots & v^j_{jn_j,jk} & \cdots & v^j_{jn_j,jn_j} \end{bmatrix}$$

have the sign of $(-1)^{q+1}$, $j = 1, \ldots, m$.

Proof. The principal minors of order q ($q \geq 3$) of the matrices C^j, $j = 1, \ldots, m$, have the sign of $(-1)^{q+1}$ as $V_j > 0$, $j = 1, \ldots, m$, by Proposition 1, and the principal minors of order q ($q \geq 3$) of the matrix A have the sign of $(-1)^{q+1}$, by Assumption 3, by Theorem 6 in Green (1964). ■

According to Assumption 4 and Proposition 2, the functions v^j are strongly increasing and strictly quasi-concave, $j = 1, \ldots, m$.

3 Expenditure minimization and homogeneity of degree one

Consider the following expenditure minimization problem

$$\begin{aligned} &\min_{\bar{x}_j} \bar{p}_j \bar{x}_j \\ &\text{subject to} \\ &v^j(\bar{x}_j) \geq v, \end{aligned} \tag{2}$$

$j = 1, \ldots, m$.

Let $v^{j0} = v^j(0, \ldots, 0)$, $j = 1, \ldots, m$.

We can now state and prove the following propositions (see also Espinosa and Prada (2012)).

Proposition 3. *Under Assumption 2, $v^{j0} = 0$, $j = 1, \ldots, m$.*

Proof. We have that

$$v^{j0} = v^j(0, \ldots, 0) = v^j(t0, \ldots, t0) = tv^j(0, \ldots, 0),$$

for each $t > 0$, as the function v^j is homogeneous of degree one, by Assumption 2. Hence, it must be that $v^{j0} = 0$, $j = 1, \ldots, m$.

Proposition 4. *$v \leq v^{j0}$ if and only if $\bar{x}_j = (0, \ldots, 0)$ is a solution to the expenditure minimization problem (2), $j = 1, \ldots, m$.*

Proof. Suppose that $v \leq v^{j0}$. Then, it must be that $\bar{x}_j = (0, \ldots, 0)$ is a solution to the expenditure minimization problem (2). Suppose now that $\bar{x}_j = (0, \ldots, 0)$ is a solution to the expenditure minimization problem (2). Moreover, suppose that $v > v^{j0}$. Then, we have that

$$v^{j0} = v^j(\bar{x}_j) = v^j(0, \ldots, 0) \geq v > v^{j0},$$

a contradiction. But then, it must be that $v \leq v^{j0}$. Hence, we have that $v \leq v^{j0}$ if and only if $\bar{x}_j = (0, \ldots, 0)$ is a solution to the expenditure minimization problem (2), $j = 1, \ldots, m$. ■

From Propositions 3 and 4, we can rewrite the expenditure minimization problem (2) as

$$\begin{aligned}
&\min_{\bar{x}_j} \bar{p}_j \bar{x}_j \\
&\text{subject to} \\
&v^j(\bar{x}_j) \geq v \geq 0,
\end{aligned} \tag{3}$$

as $v^{j0} = 0$, $j = 1, \ldots, m$.

It is well known that, under Assumptions 1, 2, 3, and 4, there exists a unique solution $\tilde{\bar{x}}_j$ to the expenditure minimization problem (3), $j = 1, \ldots, m$ (see, for instance, Exercise 3.E.3 and Proposition 3.E.3 in Mas-Colell et al. (1995)) and $v^j(\tilde{\bar{x}}_j) = v$, $j = 1, \ldots, m$ (see, for instance, Proposition 10.2 in Kreps (2013)). Therefore, we can to further specify the expenditure minimization problem (3) as follows.

$$\begin{aligned}
&\min_{\bar{x}_j} \bar{p}_j \bar{x}_j \\
&\text{subject to} \\
&v^j(\bar{x}_j) = v \geq 0,
\end{aligned} \tag{4}$$

$j = 1, \ldots, m$.

Let $e^j(\bar{p}_j, v^j)$ denote the expenditure minimization function, i.e., the function which associates with each price vector $\bar{p}_j$ and each level of utility $v \geq 0$, the unique solution to the expenditure minimization problem (4), $j = 1, \ldots, m$. It is also well known that, under Assumptions 1, 2, 3, and 4, $e^j(\bar{p}_j, v^j) = e^j(\bar{p}_j, 1)v^j$, $j = 1, \ldots, m$ (see, for instance, Corollary 1 in Espinosa and Prada (2012)). Since $e^j(\bar{p}_j, 1)$ will play the role of a price index of each subset of commodities N_j in the two-state maximization procedure, we conclude this section showing that it is strictly positive.

Proposition 5. *Under Assumptions 1, 2, 3, and 4, $e^j(\bar{p}_j, 1) > 0$, $j = 1, \ldots, m$.*

Proof. We have that $e^j(\bar{p}_j, 1) \geq 0$, $j = 1, \ldots, m$, as $\bar{p}_j \in R^{n_j}_{++}$ and $\bar{x}_j \in R^{n_j}_{+}$, $j = 1, \ldots, m$. Suppose that $e^j(\bar{p}_j, 1) = 0$ for some j. Let $\tilde{\bar{x}}_j$ be the unique solution to the expenditure minimization problem (4) for $v = 1$. Then, we have that $\tilde{\bar{x}}_j = 0$ as $\bar{p}_j \in R^{n_j}_{++}$. But then, it must be that $v^j(\tilde{\bar{x}}_j) = v^{j0} = 0$, by Propositions 3 and 4. However, it must also be that $v^j(\tilde{\bar{x}}_j) = 1$, a contradiction. Hence, we have that $e^j(\bar{p}_j, 1) > 0$, $j = 1, \ldots, m$. ■

4 Two-stage budgeting through utility maximization and expenditure minimization

Consider the following maximization problem.

$$\begin{aligned}
&\max_{v^1, \ldots, v^m \in R^m_+} V(v^1, \ldots, v^m) \\
&\text{subject to} \\
&\sum_{j=1}^{m} e^j(\bar{p}_j, 1)v^j = I.
\end{aligned} \tag{5}$$

There exists a unique solution to the utility maximization problem (5), $v^{1*}, \ldots, v^{m*}$, as the function V is strongly increasing and strictly quasi-concave by Proposition 1 (see Propositions 3.D.1 and 3.D.2 in Mas-Colell et al. (1995)).

We consider a two-stage hybrid maximization procedure whose first stage determines, through the solution of the utility maximization problem (5), the quantity index v^j of each subset of commodities N_j, given a price index which corresponds to the minimum expenditure to achieve a level of utility equal to 1 at the prices $\bar{p}_j$ of the commodities belonging to the subset N_j, and whose second stage determines, through the solution of the expenditure minimization problem (4), the quantities of the commodities belonging to each subset N_j which minimize the expenditure to achieve the utility level v^j, determined as a solution of the first stage maximization problem, at the prices $\bar{p}_j$ of the commodities belonging to the subset N_j.

The following definition characterizes the consistency of this two-stage procedure according to the conditions introduced by Green (1964).

Definition 3. *Under Assumptions 1, 2, 3, and 4, the two-stage maximization procedure constituted by the utility maximization problem (5) and the expenditure minimization problem (4) is consistent if the unique solution $v^{1*}, \ldots, v^{m*}$ to the utility maximization problem (5) and the unique solution $\bar{x}_j^*$ to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \ldots, m$, are such that $e^j(\bar{p}_j, 1)v^{j*} = \bar{p}_j \bar{x}_j^*$ and $\bar{x}_j{}^* = \hat{\bar{x}}_j$, $j = 1, \ldots, m$, where $\hat{\bar{x}}_j$, $j = 1, \ldots, m$, is the unique solution to the utility maximization problem (1).*

We can now state and prove our main theorem.

Theorem. *Under Assumptions 1, 2, 3, and 4, if the unique solution $v^{1*}, \ldots,$ v^{m*} to the utility maximization problem (5), the unique solution $\bar{x}_j^*$ to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \ldots, m$, and the unique solution $\hat{\bar{x}}_j$, $j = 1, \ldots, m$, to the utility maximization problem (1) are interior, then the two-stage maximization procedure constituted by the utility maximization problem (5) and the expenditure minimization problem (4) is consistent.*

Proof. Suppose that the unique solution $v^{1*}, \ldots, v^{m*}$ to the utility maximization problem (5), the unique solution $\bar{x}_j^*$ to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \ldots, m$, and the unique solution $\hat{\bar{x}}_j$, $j = 1, \ldots, m$, to the utility maximization problem (1) are interior. We have that $e^j(\bar{p}_j, 1)v^{j*} = e^j(\bar{p}_j, v^{j*}) = \bar{p}_j \bar{x}_j^*$ as $\bar{x}_j^*$ is the unique solution to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \dots, m$. We now adapt to our framework the proof of Theorem 4 in Green (1964). Let $\hat{\bar{x}}_j$ be the unique interior solution to the utility maximization problem (1), $j = 1, \dots, m$, and let $\hat{u} = u(\hat{\bar{x}}_1, \dots, \hat{\bar{x}}_m)$. It must be that

$$\frac{\hat{u}_{ji}}{\hat{u}_{rk}} = \frac{\bar{p}_{ji}}{\bar{p}_{rk}},$$

$j = 1, \dots, m$, $i = 1, \dots, n_j$, $r = 1, \dots, m$, $k = 1, \dots, n_k$, as $\hat{\bar{x}}_j$, $j = 1, \dots, m$, is an interior solution to the utility maximization problem (1). Let $v^{1*}, \dots, v^{m*}$ be the unique interior solution to the utility maximization problem (5) and let $V^* = V(v^{1*}, \dots, v^{m*})$. It must be that

$$\frac{V_j^*}{V_r^*} = \frac{e^j(\bar{p}_j, 1)}{e^r(\bar{p}_r, 1)},$$

$j = 1, \dots, m$, $r = 1, \dots, m$, as $v^{1*}, \dots, v^{m*}$ is an interior solution to the utility maximization problem (5). Let $\bar{x}_j^*$ be e unique solution $\bar{x}_j^*$ to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \dots, m$. It must be that

$$\frac{v_{ji}^{j*}}{v_{jk}^{j*}} = \frac{\bar{p}_{ji}}{\bar{p}_{jk}},$$

$j = 1, \dots, m$, $i = 1, \dots, n_j$, $k = 1, \dots, n_k$, as $\bar{x}_j^*$ is an interior solution to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \dots, m$. Moreover, let $u^* = u(\bar{x}_1^*, \dots, \bar{x}_m^*)$. We have to show that

$$\frac{u_{ji}^*}{u_{rk}^*} = \frac{\bar{p}_{ji}}{\bar{p}_{rk}},$$

$j = 1, \dots, m$, $i = 1, \dots, n_j$, $r = 1, \dots, m$, $k = 1, \dots, n_k$. Consider first the case where $j = r$. We have that

$$\frac{u_{ji}^*}{u_{jk}^*} = \frac{V_j^* v_{ji}^{j*}}{V_j^* v_{jk}^{j*}} = \frac{v_{ji}^{j*}}{v_{jk}^{j*}} = \frac{\bar{p}_{ji}}{\bar{p}_{jk}},$$

$i = 1, \dots, n_j$, $k = 1, \dots, n_j$. Consider now the case where $j \neq r$. We have that

$$\frac{u_{ji}^*}{u_{rk}^*} = \frac{V_j^* v_{ji}^{j*}}{V_r^* v_{rk}^{j*}} = \frac{e^j(\bar{p}_j, 1) v_{ji}^{j*}}{e^r(\bar{p}_r, 1) v_{rk}^{j*}},$$

$j = 1, \ldots, m$, $i = 1, \ldots, n_j$, $r = 1, \ldots, m$, $k = 1, \ldots, n_k$, $j \neq r$. It must be that

$$v_{ji}^{j*} = \lambda_j \bar{p}_{ji},$$

$i = 1, \ldots, n_j$, where λ_j is a Lagrange multiplier, as $\bar{x}_j^*$ is an interior solution to the expenditure minimization problem (4) when $v^j(\bar{x}_j) = v^{j*}$, $j = 1, \ldots, m$. Then, we have that

$$\sum_{i=1}^{n_j} \bar{x}_{ji}^* v_{ji}^{j*} = \lambda_j \sum_{i=1}^{n_j} \bar{p}_{ji} \bar{x}_{ji}^* = \lambda_j e^j(\bar{p}_j, v^{j*}) = \lambda_j e^j(\bar{p}_j, 1) v^{j*},$$

$j = 1, \ldots, m$. But then, we have that

$$v^{j*} = \lambda_j e^j(\bar{p}_j, 1) v^{j*},$$

by Euler's theorem, as the function v^j is homogeneous of degree one, by Assumption 2, $j = 1, \ldots, m$. Therefore, it must be that

$$e^j(\bar{p}_j, 1) v_{ji}^{j*} = \bar{p}_{ji},$$

as $\lambda_j = \frac{1}{e^j(\bar{p}_j, 1)}$, $i = 1, \ldots, n_j$, $j = 1, \ldots, m$. Thus, we have that

$$\frac{u_{ji}^*}{u_{rk}^*} = \frac{V_j^* v_{ji}^{j*}}{V_r^* v_{rk}^{j*}} = \frac{e^j(\bar{p}_j, 1) v_{ji}^{j*}}{e^r(\bar{p}_r, 1) v_{rk}^{j*}} = \frac{\bar{p}_{ji}}{\bar{p}_{rk}},$$

$j = 1, \ldots, m$, $i = 1, \ldots, n_j$, $r = 1, \ldots, m$, $k = 1, \ldots, n_k$, $j \neq r$. Combining the two cases, we have shown that

$$\frac{u_{ji}^*}{u_{rk}^*} = \frac{\bar{p}_{ji}}{\bar{p}_{rk}},$$

$j = 1, \ldots, m$, $i = 1, \ldots, n_j$, $r = 1, \ldots, m$, $k = 1, \ldots, n_k$. Therefore, we have that $\bar{x_j}^* = \hat{\bar{x}}_j$, $j = 1, \ldots, m$, as $\hat{\bar{x}}_j$, $j = 1, \ldots, m$, is the unique interior solution to the utility maximization problem (1). Hence, the two-stage maximization procedure constituted by the utility maximization problem (5) and the expenditure minimization problem (4) is consistent. ∎

5 Conclusion

In this paper, we have considered a reformulation of the two-stage procedure for the utility maximization problem of a consumer proposed by Green (1964) replacing the second stage of his procedure with a problem of expenditure minimization. We have seen that this has allowed us to specify the price indices of each group of commodities as the minimum expenditure to achieve an utility level equal to one at the prices of the commodities belonging to that group.

We have proved the consistency of out two-stage procedure under the assumption that the solutions to the optimization problems are interior. We leave for further research an investigation the possibility of extending our proof to the case of boundary solutions.

Moreover, we leave to further research the possible advantages of the application of our procedure, given the simplification it allows in pricing groups of commodities, to some fields of economic analysis, starting from the theory of monopolistic competition.

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