UNIVERSITÀ DEGLI STUDI DI UDINE
DIPARTIMENTO DI SCIENZE ECONOMICHE E STATISTICHE

On the Foundation of Monopoly in Bilateral Exchange: Some Historico-Analytical Aspects

Francesca Busetto, Giulio Codognato, Franco Donzelli, Sayantan Ghosal, Damiano Turchet

January 2024

n. 1/2024

Economia Politica e Storia Economica

On the Foundation of Monopoly in Bilateral Exchange: Some Historico-Analytical Aspects*

Francesca Busetto[†], Giulio Codognato[‡], Franco Donzelli[§], Sayantan Ghosal[¶], Damiano Turchet[‖]

January 2024

Abstract

We consider, from a historico-analytical perspective, the monopoly solution in bilateral exchange introduced by Busetto et al. (2023a), reconstructing its relationships with the previous literature on monopoly in exchange economies. In particular, we provide the conditions under which this monopoly solution coincides with that defined by Kats (1974a) and those, more restrictive, under which it has the geometric characterization proposed by Schydlowsky and Siamwalla (1966). Moreover, we establish the formal relationships between the monopoly solution introduced by Busetto et al. (2023a) and that proposed by Pareto (1896), by redefining the latter in the bilateral exchange setting.

Journal of Economic Literature Classification Numbers: D42, D51.

*For the purpose of open access, the authors have added a Creative Commons Attribution (CC BY) license to any Author Accepted Manuscript version arising from this submission.

[†]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy.

[‡]Dipartimento di Scienze Economiche e Statistiche, Università degli Studi di Udine, 33100 Udine, Italy, and Economix, UPL, Univ Paris Nanterre, CNRS, F92000 Nanterre, France.

[§]Dipartimento di Scienze e Politiche Ambientali, Università degli Studi di Milano, 20133 Milano, Italy

[¶]Adam Smith Business School, University of Glasgow, Glasgow G12 8QQ, United Kingdom.

[‖]Department of Economics, University of Warwick, CV4 7AL Coventry, United Kingdom.

1 Introduction

As far as we know, Vilfredo Pareto was the first who gave a formalized treatment of the problem of monopoly for a general pure exchange economy with any finite number of commodities, in the first volume of his *Cours d'économie politique*, published in 1896, pp. 62-71 (henceforth just Pareto (1896)). His monopoly quantity-setting solution rests on the assumption that the monopolist gets no utility from the only commodity he is endowed with, and only cares about the revenue he can obtain by selling it.

Seventy years later, Schydlowsky and Siamwalla (1966) proposed a formulation of the problem of monopoly without any mention to the previous work by Pareto (1896). In the context of a pure exchange economy, they considered a bilateral framework where one commodity is held by one trader behaving as a monopolist while the other is held by a "competitors' community." In contrast to Pareto's analysis, the monopolist desires both commodities. The authors provided a geometrical representation of the monopoly solution as the point of tangency between the monopolist's indifference curve passing through the equilibrium allocation and the offer curve of the competitors' community. They did not mention either the geometrical treatment of the monopoly problem previously given, at a very embryo stage, by Edgeworth (1881, App. V, p. 114, Fig. 5).

A few years later, Kats (1974a), again without mentioning Pareto (1896), analyzed a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity-setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and with the offer curve of the competitive traders.

Busetto et al. (2023a) provided a theoretical foundation of the monopoly solution by formalizing an explicit trading process inspired to that first sketched by Pareto (1896).

In particular, they considered the mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part. They assumed that the monopolist acts strategically, making a bid of the commodity he holds in exchange for the other commodity, while the atomless part behaves *à la* Walras; given the monopolist's bid, prices adjust to equate it to the aggregate net demand of the atomless part. Each trader belonging to the atomless part then obtains his Walrasian demand whereas the monopolist's final holding is determined as the difference between his endowment and his bid, for the commodity he holds, and as the value of his bid in terms of relative prices, for the other commodity. They defined a monopoly equilibrium as a strategy played by the monopolist, represented by a positive bid of the commodity he holds, which guarantees him to obtain, via the trading process described above, a most preferred final holding among those he can achieve through his bids.

Moreover, they adapted to their monopoly bilateral exchange context the version of the Shapley windows model used by Busetto et al. (2020): they assumed that the atomless part behaves *à la* Cournot making bids of the commodity it holds. Then, they provided a sequential reformulation of the mixed version of the Shapley windows model in terms of a two-stage game with observed actions, where the quantity-setting monopolist moves first and the atomless part moves in the second stage. This two-stage reformulation of the Shapley windows model allowed them to provide a game theoretical foundation of the quantity-setting monopoly solution as they proved that the set of the allocations corresponding to a monopoly equilibrium and the set of those corresponding to a subgame perfect equilibrium of the two-stage game coincide.

Now, the theoretical framework proposed by Busetto et al. (2023a) to define and analyse a monopoly equilibrium in bilateral exchange can be simplified, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, and compared with the standard analysis of monopoly in a partial equilibrium context. Indeed, we show that, if this assumption holds, at an allocation corresponding to a monopoly equilibrium, the utility of the monopolist is maximal in the complement of the atomless part's offer which is feasible with respect to the aggregate initial endowments, thereby providing a foundation of the monopoly solution proposed by Kats (1974a).

Moreover, we show that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and the Walrasian demand of each trader in the atomless part is differentiable, both the inverse demand function of the monopolist and the offer curve of the atomless part are differentiable. Then, we use these results to integrate the reconstruction proposed by Busetto et al. (2023a) of the characterization of a monopoly equilibrium graphically sketched by Schydlowsky and Siamwalla (1966): we confirm that it rests on a notion, the marginal revenue of the monopolist, which not only has a well-known counterpart in partial equilibrium analysis, since Cournot's original discussion of monopoly equilibrium in Chapter V of his *Recherches* (1838), but was also used by Pareto (1896) to formulate his solution to the monopoly problem in exchange economies.

Finally, we go deeper into the relationship between our analysis and that proposed by Pareto (1896), by redefining and studying this author's concept of a monopoly equilibrium in the framework of bilateral exchange and by comparing it with that introduced by Busetto et al. (2023a).

The paper is organized as follows. In Section 2, we introduce the mathematical model and we define the notion of a monopoly equilibrium. In Section 3, we characterize the monopoly equilibrium when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, providing a foundation of the monopoly solutions proposed by Schydlowsky and Siamwalla (1966) and Kats (1974a). In Section 4, we redefine and analyze Pareto's monopoly equilibrium. Finally, in Section 5, we draw some conclusions and suggest some further lines of research.

2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$.[1] Moreover, we assume that $T \setminus T_0 = \{m\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

[1] The symbol 0 denotes the origin of R^2_+ as well as the real number zero: no confusion will result.

Assumption 1. *$\mathbf{w}^i(m) > 0$, $\mathbf{w}^j(m) = 0$ and $\mathbf{w}^i(t) = 0$, $\mathbf{w}^j(t) > 0$, for each $t \in T_0$, $i = 1$ or 2, $j = 1$ or 2, $i \neq j$.*

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^2_+ \to R$ is continuous, strongly monotone, and strictly quasi-concave, for each $t \in T$.*

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $D \times F$ such that $D \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

A price vector is a non-null vector $p \in R^2_+$. Let $\mathbf{X}^0 : T_0 \times R^2_{++} \to \mathcal{P}(R^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R^2_{++}$, $\mathbf{X}^0(t, p) = \text{argmax}\{u(x) : x \in R^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in R^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}(t, p)\, d\mu : \mathbf{x}(\cdot, p)$ is integrable and $\mathbf{x}(t, p) \in \mathbf{X}^0(t, p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t, \cdot)$ is nonempty and single-valued, by Assumption 2, it is possible to define the Walrasian demand of traders in the atomless part as the function $\mathbf{x}^0 : T_0 \times R^2_{++} \to R^2_+$ such that $\mathbf{X}^0(t, p) = \{\mathbf{x}^0(t, p)\}$, for each $t \in T_0$ and for each $p \in R^2_{++}$.

We reformulate now the following proposition, proved by Busetto et al. (2023a).

Proposition 1. *Under Assumptions 1, 2, and 3, the function $\mathbf{x}^0(\cdot, p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$ for each $p \in R^2_{++}$.*

Proof. See the proof of Proposition 1 in Busetto et al. (2023a). ■

Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\ i = 1, 2\}$ denote the strategy set of atom m. We denote by $e \in \mathbf{E}(m)$ a strategy of atom m, where e_{ij}, $i, j = 1, 2$, represents the amount of commodity i that atom m offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definition.

Definition 1. *Given a strategy $e \in \mathbf{E}(m)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \int_{T_0} \mathbf{x}^{0j}(t, p)\, d\mu + \sum_{i=1}^2 e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^2 e_{ji}\mu(m), \quad (1)$$

$j = 1, 2$.

Market clearing price vectors can be normalized by Proposition 2 in Busetto et al. (2023a). Henceforth, we say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R^2_+ : \sum_{i=1}^2 p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

The next proposition shows that the two equations in (1) are not independent.

Proposition 2. *Under Assumptions 1, 2, and 3, given a strategy $e \in \mathbf{E}(m)$, a price vector $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$.*

Proof. See the proof of Proposition 3 in Busetto et al. (2023a). ■

We need to repropose now a proposition, proved by Busetto et al. (2023a), which provides a necessary and sufficient condition for the existence of a market clearing price vector. In order to state it, we introduce the following preliminary definitions.

Definition 2. *A square matrix C is said to be triangular if $c_{ij} = 0$ whenever $i > j$ or $c_{ij} = 0$ whenever $i < j$.*

Definition 3. *We say that commodities i, j stand in relation Q if $\mathbf{w}^i(t) > 0$, for each $t \in T_0$, and there is a nonnull subset T^i of T_0 such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v^i_t(x^i) + v^j_t(x^j)$, for each $x \in R^2_+$, and $\frac{dv^j_t(0)}{dx^j} = +\infty$, for each $t \in T^i$.*[2]

Moreover, we introduce the following assumption.

Assumption 4. *Commodities i, j stand in relation Q.*

Proposition 3. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. See the proof of Proposition 5 in Busetto et al. (2023a). ■

We denote by $\pi(\cdot)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is

[2] In this definition, differentiability is to be understood as continuous differentiability and includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012, p. 58)).

equal to $\{0\}$, otherwise. A price selection $p(\cdot)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned} \mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m), \text{ otherwise,} \end{aligned}$$

$j = 1, 2$,

$$\begin{aligned} \mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p), \text{ if } p \in \Delta \setminus \partial\Delta, \\ \mathbf{x}^j(t,p) &= \mathbf{w}^j(t), \text{ otherwise,} \end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a strategy $e \in \mathbf{E}(m)$ and a price selection $p(\cdot)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(m) = \mathbf{x}(m, e, p(e)),$$

$$\mathbf{x}(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$. Traders' final holdings constitute an allocation, by Proposition 6 in Busetto et al. (2023a). Moreover, it is straightforward to verify that $p(e)\mathbf{x}(m, e, p(e)) = p(e)\mathbf{w}(m)$.

We can now provide the definition of a monopoly equilibrium.

Definition 4. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_m(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_m(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

A monopoly allocation is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium, with respect to a price selection $p(\cdot)$.

3 Monopoly equilibrium and invertible aggregate demand

The analysis of the monopoly problem in bilateral exchange proposed in the previous sections can be simplified by introducing the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and compared, under this restriction, with the standard partial equilibrium analysis of monopoly.

We now show that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, our model can provide an economic theoretical foundation of the solutions proposed by Schydlowsky and Siamwalla (1966) and Kats (1974a).

We remind that, under Assumptions 1, 2, 3, and 4, if $\mathbf{w}^i(m) > 0$, then $\int_{T_0} \mathbf{x}^{0i}(t,p)\, d\mu > 0$, for each $p \in \Delta \setminus \partial\Delta$, by the argument used by Busetto et al. (2023a) in the proof of their Proposition 5.

The following proposition states a necessary and sufficient condition for the atomless part's aggregate demand to be invertible.

Proposition 4. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$. Then, the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible on R_{++} if and only if, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t,p)\, d\mu$.*

Proof. See the proof of Proposition 7 in Busetto et al. (2023a). ■

Suppose that $\mathbf{w}^i(m) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible on R_{++}. Let $p^{0i}(\cdot)$ denote the inverse of the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ and let $\mathring{p}(e)$ be a function which associates, with each strategy $e \in \mathbf{E}(m)$, the price vector $p = p^{0i}(e_{ij}\mu(m))$, if E is triangular, and is equal to $\{0\}$, otherwise. Then, $\mathring{p}(\cdot)$ is the unique price selection as $\pi(e) = \{\mathring{p}(e)\}$, for each $e \in \mathbf{E}(m)$. By analogy with partial equilibrium analysis, $\mathring{p}(\cdot)$ can be called the inverse demand function of the monopolist.

When the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the monopoly equilibrium in Definition 4 can be reformulated with respect to monopolist's inverse demand function $\mathring{p}(\cdot)$. Moreover, under this assumption, the monopoly equilibrium can be characterized by means of the notion of offer curve of the atomelss part, defined as the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$, and that of the notion of feasible complement of the offer curve of the atomless part, defined as the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$.

The following proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the feasible complement of the atomless part's offer curve is a subset of the set of the monopolist's final holdings.

Proposition 5. *Under Assumptions 1, 2, 3, and 4, if $\mathbf{w}^i(m) > 0$ and* *the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\,d\mu$ *is invertible on* R_{++}*, then the feasible complement of the offer curve of the atomless part, the set* $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\,d\mu = \int_T \mathbf{w}(t)\,d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$*, is a subset of the set* $\{x \in R^2_+ : x = \mathbf{x}(m, e, \mathring{p}(e))$ *for some* $e \in \mathbf{E}(m)\}$*, the set of the final holdings of the monopolist.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\,d\mu$ is invertible on R_{++}. Consider a commodity bundle $\bar{x} \in \{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\,d\mu = \int_T \mathbf{w}(t)\,d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Suppose that $\bar{x}^1 = \mathbf{w}^1(m)$. Then, we have that $\int_{T_0} \mathbf{x}^{0i}(t,p)\,d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$, a contradiction. But then, it must be that $0 \leq \bar{x}^1 < \mathbf{w}^1(m)$. Let $\bar{e} \in \mathbf{E}(m)$ be such that $\bar{e}_{12} = \mathbf{w}^1(m) - \bar{x}^1$ and let $\bar{p} = \mathring{p}(\bar{e})$. Then, we have that

$$\begin{aligned}&\bar{x}^1\mu(m) + \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\,d\mu \\ &= (\mathbf{w}^1(m) - \bar{e}_{12})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\,d\mu = \mathbf{w}^1(m)\mu(m),\end{aligned}$$

as $\bar{p} = \mathring{p}(\bar{e})$. Moreover, $\bar{p}$ is the unique price vector such that

$$(\mathbf{w}^1(m) - \bar{x}^1)\mu(m) = \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\,d\mu,$$

as the function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\,d\mu$ is invertible. Then, it must be that

$$\bar{x}^2\mu(m) + \int_{T_0} \mathbf{x}^{02}(t,\bar{p})\,d\mu = \int_{T_0} \mathbf{w}^2(t)\,d\mu,$$

by Proposition 2. But then, we have that

$$\bar{x}^2 = e_{12}\frac{\bar{p}^1}{\bar{p}^2},$$

as $\bar{p}$ is market clearing. Therefore, we conclude that

$$\bar{x} = \mathbf{x}(m, \bar{e}, \bar{p}) = \mathbf{x}(m, \bar{e}, \mathring{p}(\bar{e})).$$

Hence, the feasible complement of the offer curve of the atomless part, the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\,d\mu = \int_T \mathbf{w}(t)\,d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R^2_+ : x = \mathbf{x}(m, e, \mathring{p}(e))$ for some $e \in \mathbf{E}(m)\}$, the set of the final holdings of the monopolist. ■

Let $h(\cdot)$ be a function, defined on R_{++}, such that

$$p^i x^i + p^j x^j = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu, \tag{2}$$

where $p = p^{0i}(x^i)$ and $x^j = h(x^i)$.

The next proposition shows that $h(\cdot)$ represents the offer curve of the atomless part in the sense that its graph coincides with the atomless part's offer curve.

Proposition 6. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible on* R_{++} *, then the graph of the function* $h(\cdot)$*, the set* $\{x \in R^2_+ : x^j = h(x^i)\}$*, coincides with the set* $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$*, the offer curve of the atomless part.*

Proof. See the proof of Proposition 9 in Busetto et al. (2023a). ■

Kats (1974a) considered both the cases of a quantity setting and a price setting monopoly in a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity setting solution must correspond to the monopolist's most preferred commodity bundle compatible with both the aggregate initial endowments and the offer curve of the competitive traders. However, he did not propose any explicit trading process which could lead to the monopoly solution. The following proposition, which follows from Proposition 5, establishes that, at a monopoly allocation, the utility of the monopolist is maximal in the feasible complement of the atomless part's offer curve. Thereby, it provides an explicit economic theoretical foundation of the monopoly solution proposed by Kats (1974a).

Proposition 7. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(m) > 0$*, the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible on* R_{++}*, and* $\tilde{e} \in \mathbf{E}(m)$ *is a monopoly equilibrium, then* $u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e})))$ *is maximal in the set* $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t, p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$.

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that

$$\begin{aligned} &\mathbf{x}^1(m, \tilde{e}, \tilde{p})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \tilde{p})\, d\mu \\ &= (\mathbf{w}^1(m) - \tilde{e}_{12})\mu(m) + \int_{T_0} \mathbf{x}^{01}(t, \tilde{p})\, d\mu = \mathbf{w}^1(m)\mu(m), \end{aligned}$$

and

$$\mathbf{x}^2(m,\tilde{e},\tilde{p})\mu(m) + \int_{T_0} \mathbf{x}^{02}(t,\tilde{p})\, d\mu$$
$$= \tilde{e}_{12}\mu(m)\frac{\tilde{p}^1}{\tilde{p}^2} + \int_{T_0} \mathbf{x}^{02}(t,\tilde{p})\, d\mu = \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

as $\tilde{p}$ is market clearing. Then, we have shown that $\mathbf{x}(m,\tilde{e},\mathring{p}(\tilde{e})) \in \{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. But then, we have that $u_m(\mathbf{x}(m,\tilde{e},\mathring{p}(\tilde{e})))$ is maximal in the set $\{x \in R^2_+ : x\mu(m) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$ as $u_m(\mathbf{x}(m,\tilde{e},\mathring{p}(\tilde{e})) \geq u_m(\mathbf{x}(m,e,\mathring{p}(e))$, for each $e \in \mathbf{E}(m)$, and $\{x \in R^2_+ : x\mu(m)+\int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta\setminus\partial\Delta\} \subset \{x \in R^2_+ : x = \mathbf{x}(m,e,\mathring{p}(e))$ for some $e \in \mathbf{E}(m)\}$, by Proposition 6. ■

We now reproduce and integrate the reconstruction proposed by Busetto et al. (2023a) of the characterization of a monopoly equilibrium graphically sketched by Schydlowsky and Siamwalla (1966). In particular, we show that, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, the monopoly equilibrium introduced in Definition 4 has also the geometric characterization previously proposed by Schydlowsky and Siamwalla (1966): at a strictly positive monopoly allocation, the monopolist's indifference curve is tangent to the atomless part's offer curve.[3]

In the rest of this section, with a slight abuse of notation, given a price vector $(p^i, p^j) \in \Delta\setminus\partial\Delta$, we denote by p the scalar $p = \frac{p^i}{p^j}$, whenever $\mathbf{w}^i(m) > 0$.

By means of the following proposition, we then show that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and the Walrasian demand of traders in the atomless part is differentiable, the inverse demand function of the monopolist is differentiable (see also Proposition 4 in Busetto et al. (2023b)).

Proposition 8. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t,\cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Then, the function $\mathring{p}(\cdot)$ is differentiable at each $e \in \mathbf{E}(m)$ such that E is triangular.*

[3]This characterization of the monopoly equilibrium has been diffusely reproposed in standard textbooks in microeconomics (see, for instance, Varian (2014, p. 619), among others).

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}, and the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$. We have that

$$p\mathbf{x}^{01}(t,p) + \mathbf{x}^{02}(t,p) = p\mathbf{w}^1(t) + \mathbf{w}^2(t),$$

for each $t \in T_0$ and for each $p \in R_{++}$, as $u_t(\cdot)$ is strongly monotone, for each $t \in T_0$, by Assumption 2. Differentiating with respect to p, we obtain

$$\mathbf{x}^{01}(t,p)dp + p\frac{d\mathbf{x}^{01}(t,p)}{dp}dp + \frac{d\mathbf{x}^{02}(t,p)}{dp}dp = \mathbf{w}^1(t)dp.$$

Then, we have that

$$\frac{d\mathbf{x}^{01}(t,p)}{dp} = \frac{\mathbf{w}^1(t) - \mathbf{x}^{01}(t,p) - \frac{d\mathbf{x}^{02}(t,p)}{dp}}{p} \leq \frac{\mathbf{w}^1(t)}{p}.$$

But then, the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is differentiable on R_{++} and

$$\frac{d\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu}{dp} = \int_{T_0} \frac{d\mathbf{x}^{01}(t,p)}{dp}\, d\mu,$$

for each $p \in R_{++}$, as the function $\mathbf{x}^0(\cdot, p)$ is integrable, for each $p \in R_{++}$, by Proposition 1, and the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$, by Theorem 6.26 in Klenke (2020). Therefore, the function $p^{01}(\cdot)$ is differentiable on R_{++} as $\frac{dp^{01}(x^1)}{dx^1} = (\frac{d\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu}{dp})^{-1}$, by the inverse function theorem. Hence, we have that the function $\mathring{p}(\cdot)$ is differentiable as $\frac{d\mathring{p}(e)}{de_{12}} = \frac{dp^{01}(e_{12}\mu(m))}{d(x^1)}\mu(m)$, at each $e \in \mathbf{E}(m)$ such that E is triangular. ∎

Borrowing from Pareto (1896), we now introduce in our general framework a notion which has a counterpart in partial equilibrium analysis: the marginal revenue of the monopolist.

We know, from Proposition 7, that, when $\mathbf{w}^i(m) > 0$, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, and the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and we have that $\frac{d\mathring{p}(e)}{de_{ij}} = \frac{dp^{0i}(e_{ij}\mu(m))}{dx^i}\mu(m)$, at each $e \in \mathbf{E}(m)$ such that E is triangular. In this context, the revenue of the monopolist can be defined as $\mathring{p}(e)e_{ij}$ and his marginal revenue as $\frac{d\mathring{p}(e)}{de_{ij}}e_{ij} + \mathring{p}(e)$, for each $e \in \mathbf{E}(m)$ such that E is triangular.

The following proposition is a straightforward consequence of Proposition 8, proving that, under the same assumptions, the function $h(\cdot)$ is differentiable.

Proposition 9. *Under Assumptions 1, 2, 3, and 4, let* $\mathbf{w}^i(m) > 0$ *and let the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *be invertible on* R_{++} *and the function* $\mathbf{x}^0(t, \cdot)$ *be differentiable on* R_{++}*, for each* $t \in T_0$*. Then, the function* $h(\cdot)$ *is differentiable on* R_{++}*.*

Proof. Assume that $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. From (2), we have that

$$h(x^i) = -p^{0i}(x^i)x^i + \int_{T_0} \mathbf{w}^j(t)\, d\mu.$$

Hence, the function $h(\cdot)$ is differentiable on R_{++} as the function $p^{01}(\cdot)$ is differentiable on R_{++}, by the argument used in the proof of Proposition 8.

■

In order to provide the characterization of a monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966), we need to introduce also the following assumption.

Assumption 5. $u_m : R^2_+ \to R$ *is differentiable.*

Then, in the next proposition, which is a reformulation of Proposition 10 in Busetto et al. (2023a), we can provide a formal foundation of the geometric characterization of the monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966). Indeed, our proposition establishes that, at an interior monopoly solution, both the slope of the monopolist's indifference curve and the slope of the atomless part's offer curve are equal to the opposite of the monopolist's marginal revenue. Therefore, the tangency characterization of a monopoly equilibrium is demonstrated.

Proposition 10. *Under Assumptions 1, 2, 3, 4, and 5, if* $\mathbf{w}^i(m) > 0$, $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible on* R_{++}*, the function* $\mathbf{x}^0(t, \cdot)$ *is differentiable on* R_{++}*, for each* $t \in T_0$*, and* $\tilde{e} \in \mathbf{E}(m)$ *is a monopoly equilibrium such that* $\tilde{e} < \mathbf{w}^i(m)$*, then*

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m)}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i},$$

where $\tilde{\mathbf{x}}$ *is the monopoly allocation corresponding to* $\tilde{e}$*.*

Proof. It straightforwardly follows by adapting *mutatis mutandis* the proof of Proposition 10 in Busetto et al. (2023a). ■

Finally, we entirely repropose an example provided by Busetto et al. (2023a), that illustrates the geometric characterization of a monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966) in order to facilitate comparison with the Pareto monopoly solution discussed in the next section.

Example 1. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4.* $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, T_0 *is taken with Lebesgue measure,* $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x^1} + x^2$, *for each* $t \in T_0$*. Then, at the unique monopoly equilibrium* $\tilde{e} \in \mathbf{E}(m)$*, the slope of the indifference curve of the monopolist is equal to the opposite of his marginal revenue, which, in turn, is equal to the slope of the function which represents the offer curve of the atomless part.*

Proof. The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{e}_{12} = \frac{1}{4}$, $\mathring{p}(\tilde{e}) = 1$, $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$, and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$. Moreover, we have that $x^2 = h(x^1) = -\frac{\sqrt{x}^1}{2} + 1$ and

$$-\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^i}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = -\frac{1}{2} = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}.$$

■

4 Discussion of the literature

In the bilateral monopolistic framework of Shitovitz' Example 1 (1973, pp. 486-487), Aumann (1973) provided three examples, which show that monopoly may be, according to his terminology, "disadvantageous." In fact, let us provisionally subscribe, for discussion's sake, to Shitovitz' assumption that in a "monopolistic market" a core allocation can suitably represent the market outcome, as regards the monopolist, too. Under such assumption, Aumann's examples are able to refute the following "conjecture," implicit in Shitovitz' reasoning, which embodies the idea that a monopolist can always gain a definite advantage over competitive traders: "Conjecture. In a monopolistic market, for each core allocation $\mathbf{x}$ there is a competitive allocation $\mathbf{y}$ whose utility to the monopolist is $\leq$ that of $\mathbf{x}$" (see p. 1).

Kats (1974b) quoted the following passage from Aumann (1973) in which this author explained what is wrong in the preceding "conjecture" in spite of its intuitive appeal: "One feels on an intuitive, common sense level that the monopolist has a distinct advantage; but economic theory, rather than explaining this phenomenon, simply states it in a specific form. For an *explanation*, one looks to game theory; but evidently, the game-theoretic notion of core is not the proper vehicle for such an explanation" (see p. 10).

Kats (1974b) explicitly addressed the problem raised by Aumann (1973), proposing a variant of the model of strategic exchange introduced by Debreu (1952) in which only the monopolist is allowed to manipulate the strategy sets of other traders. Nevertheless, we can apply to this model the same criticism raised by Shapley and Shubik (1977) at the model proposed by Debreu (1952): "[...] as a *descriptive* model his game shares the defect of the Walrasian model of being ill defined, or unrealistically defined, away from equilibrium" (see p. 939, fn. 1). For this reason, we have focused here on the monopoly solution proposed by Kats (1974a), which perfectly fits in the model of monopoly proposed by Busetto et al. (2023a), since this model has a game-theoretical foundation.

Turchet (2023) started to investigate the issue of the existence of a monopoly equilibrium under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and that traders belonging to the atomless part have an identical CES utility function. In this paper, we do not provide a proof of the existence of a monopoly equilibrium, but we use the framework considered by Turchet (2023) in order to assess the role of the assumptions we have made in Section 2 to guarantee that an existence result can be reached. In particular, Assumption 1 guarantees the pure monopoly nature of the economy, while Assumption 3 is a standard measurability assumption extended to mixed exchange economies by Shitovitz (1973). These assumptions must be maintained to assure the basic economic and mathematical consistency of the monopoly model. On the other hand, the following example shows that Assumptions 2 and 4 cannot be weakened or omitted without affecting the existence of monopoly equilibrium. The example exhibits an exchange economy where traders in the atomless part have continuous, monotone, and quasi-concave utility functions that do not satisfy Assumptions 2 and 4: in this case, a monopoly equilibrium does not exist.

Example 2. *Consider the following specification of an exchange economy satisfying Assumptions 1 and 3.* $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = \sqrt{x^1} + \sqrt{x^2}$, T_0 *is taken with Lebesgue measure,* $\mathbf{w}(t) = (0,1)$, $u_t(x) = \min\{x^1, x^2\}$, *for each* $t \in T_0$. *Then,* $u_m(\cdot)$ *satisfies Assumption 2,* $u_t(\cdot)$ *is continuous, monotone, and quasi-concave, for each*

$t \in T_0$, the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible, $\mathring{p}(e) = (1 - e_{12}, e_{12})$, and there is no monopoly equilibrium.

Proof. It is straightforward to verify that $u_m(\cdot)$ satisfies Assumption 2 and that $u_t(\cdot)$ is continuous, monotone, and quasi-concave. The function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible as $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu = p_2$. Moreover, it is immediate to verify that $\mathring{p}(e) = (1 - e_{12}, e_{12})$. Suppose that the strategy $\tilde{e} \in \mathbf{E}(m)$ is a monopoly equilibrium. Then, we have that $\tilde{e} > 0$ and $\mathbf{x}(m, \tilde{e}, p(\tilde{e})) = (1 - \tilde{e}_{12}, 1 - \tilde{e}_{12})$. Let $e' \in \mathbf{E}(m)$ be a strategy such that $0 < e'_{12} < \tilde{e}_{12}$. Then, we have that

$$\begin{aligned} u_m(\mathbf{x}(m, e', p(e')) &= \sqrt{1 - e'_{12}} + \sqrt{1 - e'_{12}} \\ &> \sqrt{1 - \tilde{e}_{12}} + \sqrt{1 - \tilde{e}_{12}} = u_m(\mathbf{x}(m, \tilde{e}, p(\tilde{e})), \end{aligned}$$

a contradiction. Hence there is no monopoly equilibrium ■

Pareto (1986) was the first author who gave a formalized treatment of the problem of monopoly for a general pure exchange economy. To better understand the relationship between the analysis developed in the previous sections and that proposed by Pareto (1896), we reformulate now this author's monopoly solution within our framework of bilateral exchange.

Pareto (1896) assumed that, for the monopolist, the commodity he is endowed with is "neutral," i.e., it is a commodity from which he does not get any utility.[4] To incorporate this assumption in our model, we amend Assumption 2 as follows.

Assumption 2′. *$u_m(x) = x^j$, whenever $\mathbf{w}^i(m) > 0$, $i \neq j$, and $u_t : R^2_+ \to R$ is continuous, strongly monotone, strictly quasi-concave, for each $t \in T_0$.*

It is straightforward to verify that Assumption 2′ implies that the utility function of the monopolist is continuous, monotone, and quasi-concave.

Hereafter, we assume that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, whenever $\mathbf{w}^i(m) > 0$. Therefore, the revenue of the monopolist can be defined again as $\mathring{p}(e)e_{ij}$.

According to Pareto (1896), the goal of the monopolist is to maximize his revenue. Therefore, we can provide the following definition of a Pareto monopoly equilibrium.

[4] For a discussion of the properties of neutral commodities, see, for instance, Varian (2014).

Definition 5. *Let $\mathbf{w}^i(m) > 0$. A strategy $\hat{e} \in \mathbf{E}(m)$ such that $\hat{E}$ is triangular is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$, if*

$$\mathring{p}(\hat{e})\hat{e}_{ij} \geq \mathring{p}(e)e_{ij},$$

for each $e \in \mathbf{E}(m)$.

A Pareto monopoly allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(m) = \mathbf{x}(m, \hat{e}, \mathring{p}(\hat{e}))$ and $\hat{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\hat{e}))$, for each $t \in T_0$, where $\hat{e}$ is a Pareto monopoly equilibrium.

The following proposition shows that, when Assumption 2 is replaced with Assumption 2′, a strategy of the monopolist is a Pareto monopoly equilibrium if and only if it is a monopoly equilibrium. Moreover, it shows that, if the inverse demand function of the monopolist is differentiable, then at a Pareto monopoly solution the monopolist's marginal revenue must be nonnegative.

Proposition 11. *Under Assumptions 1, 2′, 3, and 4, let $\mathbf{w}^i(m) > 0$. Then, a strategy $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, if and only if it is a monopoly equilibrium, with respect to the same price selection. Moreover, if the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}, the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$, and $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, then*

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}}\hat{e}_{ij} + \mathring{p}(\hat{e}) \geq 0.$$

Proof. Let $\mathbf{w}^i(m) > 0$. Suppose that the strategy $\hat{e} \in \mathbf{E}(m)$ is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. Then, we have that

$$\mathring{p}(\hat{e})\hat{e}_{ij} \geq \mathring{p}(e)e_{ij},$$

for each $e \in \mathbf{E}(m)$. But then, it must be that

$$u_m(\mathbf{x}(m, \hat{e}, \mathring{p}(\hat{e})) \geq u_m(\mathbf{x}(m, e, \mathring{p}(e)),$$

for each $e \in \mathbf{E}(m)$, as

$$u_m(\mathbf{x}(m, e, \mathring{p}(e)) = \mathring{p}(e)e_{ij},$$

by Assumption 2′, for each $e \in \mathbf{E}(m)$. Therefore, the strategy $\hat{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. The converse can be straightforwardly proved by the same argument. Hence, a strategy $\hat{e} \in E(m)$ is a Pareto monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$, if and only if it is a monopoly equilibrium, with respect to the same price selection. Suppose that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}, the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$. Let $\hat{e} \in \mathbf{E}(m)$ be a Pareto monopoly equilibrium. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable at each $e \in \mathbf{E}(m)$ such that E is triangular, by Proposition 8, and the necessary Kuhn-Tucker conditions imply that

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}} \hat{e}_{ij} + \mathring{p}(\hat{e}) \geq 0.$$

■

We now provide an example of a Pareto monopoly equilibrium.

Example 3. *Consider the following specification of an exchange economy satisfying Assumptions 1, $2'$, 3, 4. $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1,0)$, $u_m(x) = x^2$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0,1)$, $u_t(x) = \sqrt{x^1} + x_2$, for each $t \in T_0$. Then, there is a unique Pareto monopoly equilibrium $\hat{e} \in \mathbf{E}(m)$ such that*

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}} \hat{e}_{ij} + \mathring{p}(\hat{e}) > 0.$$

Proof. The unique Pareto monopoly equilibrium is the strategy $\hat{e}$ such that $\hat{e}_{12} = 1$, $\mathring{p}(\hat{e}) = \frac{1}{2}$, $\hat{\mathbf{x}}(m) = (0, \frac{1}{2})$, $\hat{\mathbf{x}}(t) = (1, \frac{1}{2})$, for each $t \in T_0$. Moreover, we have that

$$\frac{d\mathring{p}(\hat{e})}{de_{ij}} \hat{e}_{ij} + \mathring{p}(\hat{e}) = \frac{1}{4}.$$

■

Comparing the monopoly solution of Example 1 with the Pareto monopoly solution of Example 3, we can observe that the atomless part is better off at the Pareto monopoly solution than at the monopoly solution as

$$u_t(\hat{\mathbf{x}})(t) = \frac{3}{2} > \frac{5}{4} = u_t(\tilde{\mathbf{x}})(t),$$

for each $t \in T_0$.

Moreover, Example 3 shows that when the utility function of the monopolist is continuous, monotone, and quasi-concave a monopoly equilibrium may exist whereas Example 2 showed that this is not the case when the assumptions made therein, weaker than those imposed by Assumption 2, hold for the atomless part.

5 Conclusion

In this paper, we have considered the model of monopoly in bilateral exchange introduced by Busetto et al. (2023a), and we have reconstructed, with a historico-analytical approach, its relationships with the previous contributions on monopoly proposed in the literature. We have first shown that, under the assumption that the aggregate demand of the atomless part of the economy for the commodity held by the monopolist is invertible, the monopoly solution introduced by Busetto et al. (2023a) coincides with that proposed by Kats (1974a). We have then shown that, if the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and the Walrasian demand of traders in the atomless part is differentiable, that same monopoly solution has the geometric characterization proposed by Schydlowsky and Siamwalla (1966), and we have provided an example of this configuration. Finally, we have studied the relationships between the monopoly solution introduced by Busetto et al, (2023a) and that proposed in the pioneering work by Pareto (1896). To do so, we have redefined his equilibrium notion, originally proposed for a general exchange economy with any finite number of commodities, in the bilateral exchange framework.

Here, we have considered a quantity-setting monopolist. We leave for future research addressing the problem of a price-setting monopolist, in the same bilateral framework as that used in this paper, drawing inspiration from another pioneering work by Vilfredo Pareto (see Pareto (1909, pp. 210-211, 594-605)).

Kats (1974a), in his final remarks (see p. 31), raised the question of the relationship between monopoly equilibrium and cooperative game theory. He formalized a monopolistic market game based on the notion of a monopolistic quasi-core and referred that to Shitovitz (1973) as the only other contribution which had addressed a similar issue, using cooperative game theory. Shitovitz (1973), in his Example 1, actually showed that, in the mixed version of a monopolistic two-commodity exchange economy, the set of allocations in the core does not coincide with the set of Walrasian allocations. Busetto et al. (2023b) referred to the debate following from Example 1 in Shitovitz (1973) to formalize and discuss the welfare properties of the monopoly solution introduced by Busetto et al. (2023a). We leave for further research also a discussion of the welfare properties of the monopoly solution proposed by Kats (1974a) with reference to his notion of monopolistic quasi-core.

References

[1] Aumann R.J. (1969), "Measurable utility and measurable choice theorem," in Guilbaud G.T. (ed), *La Décision*, Editions du Centre National de la Recherche Scientifique, Paris.

[2] Aumann R.J. (1973), "Disadvantageous monopolies," *Journal of Economic Theory* **6**, 1-11.

[3] Busetto F., Codognato G., Ghosal S., Julien L., Tonin S. (2020), "Existence and optimality of Cournot-Nash equilibria in a bilateral oligopoly with atoms and an atomless part," *International Journal of Game Theory* **49**, 933-951.

[4] Busetto F., Codognato G., Ghosal S., Turchet D. (2023a), "On the foundation of monopoly in bilateral exchange," *International Journal Game Theory*, **52**, 1261-1290.

[5] Busetto F., Codognato G., Ghosal S., Turchet D. (2023b), "Atitrust in the Edgeworth box: monopoly," WP n.2 2023, Dipartimento di Scienze Economche e Statistiche, Università degli Studi di Udine.

[6] Cournot A.A. (1838), *Recherches sur les principes mathématiques de la théorie des richesses*, Hachette, Paris.

[7] Debreu G. (1952), "A social equilibrium existence theorem," *Proceedings of the National Academy of Sciences of the United States of America* **38**, 886-893.

[8] Edgeworth F.Y. (1881), *Mathematical psychics*, Kegan Paul, London.

[9] Kats A. (1974a), "Monopolistic trading economies: a case of governmental control," *Public Choice* **20**, 17-32.

[10] Kats A. (1974b), "Non-cooperative monopolistic games and monopolistic market games," *International Journal of Game Theory* **3**, 251-260.

[11] Klenke A. (2020), Probability theory, Springer, New York.

[12] Kreps D. (2012), *Microeconomic foundations I: choice and competitive markets*, Princeton University Press, Princeton.

[13] Pareto V. (1896), *Cours d'économie politique*, F. Rouge Editeur, Lausanne.

[14] Pareto V. (1909), *Manuel d'économie politique*, V. Giard & E. Brière, Paris.

[15] Shapley L.S., Shubik M. (1977), "Trade using one commodity as a means of payment," *Journal of Political Economy* **85**, 937-968.

[16] Schydlowsky D.M., Siamwalla A. (1966), "Monopoly under general equilibrium: a geometric exercise," *Quarterly Journal of Economics* **80**, 147-153.

[17] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

[18] Turchet D. (2023), "Monopoly equilibrium and elasticity of substitution: a note on the existence of equilibrium," mimeo, University of Warwick.

[19] Varian H.R. (2014), *Intermediate microeconomics with calculus*, Norton, New York.