UNIVERSITÀ DEGLI STUDI DI UDINE
DIPARTIMENTO DI SCIENZE ECONOMICHE E STATISTICHE

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August 2023

n. 2/2023

Economia Politica e Storia Economica

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August 2023

Abstract

What is the appropriate economic welfare standard for antitrust in general equilibrium? In this paper, we address this question in an Edgeworth box economy model of monopoly introduced by Busetto et al. (2023), where one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part. In this framework, we reconcile the different approaches characterizing the so-called Chicago School, on one hand, and the so-called New Brandeis School, on the other. Moreover, we readapt to our context a paradox, first formulated by Shitovitz (1997), which shows that the Brandeisian "curse of bigness" overwhelms the direct exercise of market power by the monopolist.
Journal of Economic Literature Classification Numbers: D42, D51, L41.

1 Introduction

Judge Robert H. Bork, one of the leading exponent of the so-called Chicago School, provided a foundation of antitrust policy arguing that its goal should

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be “allocative efficiency.” In his celebrated book (see Bork (1978)), he considered the notion of allocative efficiency as related to that of “consumer welfare” (see p. 91). His viewpoint is at the origin of a long-standing antitrust controversy, reconstructed by Salop (2010), regarding the economic welfare standard for antitrust. According to Salop, the main point of contention was between some commentators who favored the “aggregate economic welfare” standard, sometimes called also “efficiency” standard, and other commentators who favored the “true consumer welfare” standard, sometimes called also the “pure consumer welfare” standard. Salop made precise that he introduced the “true” qualifier just because of the confusion that had resulted from Judge Robert Bork’s usage of the term “consumer welfare” in referring to aggregate welfare (see p. 336).

Lande (1982) criticized Bork’s foundation of antitrust based on allocative efficiency – which he reduced to Pareto optimality – arguing that the redistributive effects of monopoly on consumers, whereby consumers are poorer but the monopolist richer, in general exceed any effects of allocative inefficiency by a substantial amount (see pp. 74-75). After Lande’s contribution, the controversy regarding which economic welfare standard should be used for assessing the social cost of monopoly reduced to the juxtaposition between the Pareto optimality criterion and the (true) consumer welfare criterion. An aspect of the analysis that contributed to exacerbate the controversy concerned the way in which the social cost of monopoly was measured. As stressed by Brown and Lee (2008), the most familiar measure was represented by the deadweight loss triangle, originally introduced within a partial equilibrium analysis. This measure and its generalizations proposed by the Chicago School were not immune from serious ambiguities. Brown and Lee (2008) noticed that these ambiguities were principally due to the limits of partial equilibrium analysis.

They consequently proposed to use a general equilibrium theory of monopoly. Nevertheless, on this point they acknowledged the following difficulty: “The conspicuous absence of general equilibrium theory to antitrust law is due in part to the indeterminacy of the price level in the Arrow-Debreu model. As such, the model does not admit price-setting, profit maximizing firms” (see footnote 15, p. 55). This is a well-known problem of general equilibrium models with imperfect competition (see, for instance, Grodal (1996)). Anyhow, these authors affirmed: “The notion of efficiency and welfare in general equilibrium theory is Pareto optimality, also known as allocative efficiency” and, referring to Judge Bork’s analysis, they stated: “The primary goals of antitrust are efficiency and enhancing of consumer welfare. Both of these concepts appeal to Pareto optimality" (see p. 56).

Brown and Lee (2008) observed that, in the partial equilibrium analysis of the social cost of monopoly measured by deadweight loss, the relevant benchmark of allocative efficiency is the Pareto optimal state of perfect competition. Nevertheless, they further pointed out the reason why the deadweight loss analysis of the social cost of monopoly may be problematic: "[...] It ignores the social cost of inducing perfect competition [...] in a given industry, and thus assumes a counterfactual that is not attainable even by a benevolent social planner" (see p. 48).

These authors were, thus, confronted with the cumbersome puzzle consisting in determining the social cost of monopoly in terms of Pareto optimality without referring to the usual benchmarks of partial equilibrium analysis obtained by means of profit maximization: the monopoly solution and the perfectly competitive solution. They dealt with this problem by building a to some extent hybrid model they described as follows: "[...] Firms with monopoly power have unspecified price-setting rules for output [...] but are assumed to be cost-minimizing price-takers in competitive factor markets. [...] Meanwhile, in equilibrium they make supra-competitive profits since the monopoly price exceeds the marginal cost of production. Our analysis derives from a subtle but important distinction between price-setting profit-maximization [...] and monopoly power, i.e., the power to raise price above the competitive level and make supra-competitive profits" (see p. 60). Then, they presented what they considered the most appropriate benchmark for measuring the social cost of monopoly as follows: "We propose the unique Pareto optimal state characterized by Debreu's coefficient of resource utilization [...]. This coefficient is defined as the smallest fraction of total resources capable of providing consumers with utility levels at least as great as those attained in the monopolized economic state" (see p. 57).

The main goal of this paper is to establish the relation between economic welfare standard for antitrust and explicit monopoly and perfectly competitive solutions in a general equilibrium framework. Since, to the best of our knowledge, there is no general equilibrium model of monopoly with production in which the indeterminacy of profit maximization mentioned above has been overcome, we recast the issue in the simplest and tersest version of a pure exchange economy: an Edgeworth box economy.

We consider the mixed version of the monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1. There, one commodity is held only by the monopolist, represented as an atom, and the other is held only by small traders, represented by an atomless part.

As observed by Aumann (1973), in that framework, the monopolist is characterized both as a big trader, since it is represented as an atom, and the only possible seller of the commodity he holds, since he has a corner on that commodity (see p. 2). Shitovitz (1973) himself initiated an analysis of the welfare properties of monopoly in terms of the notion of core within the exchange economy sketched above. Since all allocations in the core are Pareto optimal, so that allocative efficiency is guaranteed, and they are not determined at an explicit quantity-setting or price-setting solution, the only possible benchmark reduces to the Walras equilibrium solution.

In his Example 1, Shitovitz showed that the unique Walras allocation is worse, in terms of the monopolists utility, than any other allocation in the core. This led him to formulate the following open problem: "In a market with exactly one large trader, is it true that at every allocation in the core, the large trader is not worse off in terms of utility than at the competitive equilibrium which is worst for him?" (see p. 488).

In the bilateral monopolistic framework of Shitovitz' Example 1, Aumann (1973) gave a negative answer to this question through three examples, which show that monopoly may be, according to his terminology, "disadvantageous." In the following passage, the author explains why he considered these examples counterintuitive: "According to the classical theory, the oceanic traders in a monopoly will act like price takers, i.e., they will maximize their utility given the prices set by the monopolist. The monopolist will set the prices so that the result of price-taking on the part of the ocean will maximize his utility" (see p. 9). He complained about the lack of a general foundation for such a theory while recognizing that the core fails to display the monopolist's power (see p. 10).[1] Moreover, he expressed the following desiderata: "What one would like is a theory that is applicable in any market, and when applied to a monopoly, yields the price-taking mechanism" (see p. 10).

The model of monopoly introduced by Busetto et al. (2023) provided

[1] The same criticism to the core solution was addressed by Okuno et al. (1980) referring to the Shitovitz general model of a mixed exchange economy. Indeed, they argued: "While Shitovitz model would seem especially appropriate for studying oligopoly, he concentrated not so much on market power per se as on the possibility that all the core outcomes would still be competitive allocations despite the presence of atoms. Some of the results he obtained in studying this issue appear so counterintuitive as to seem to call into question the use of this model with atoms and a nonatomic ocean in studying oligopoly. [...] if one finds such a result unsatisfying, one need not question the model of atoms and a continuum. Rather, one might object to the use of the core as the solution concept" (see p. 22).

a first answer to these desiderata, within the same setup as of Shitovitz' Example 1. They assumed that the monopolist acts strategically, making a bid of the commodity he holds in exchange for the other commodity, while the atomless part behaves *à la* Walras; given the monopolists bid, prices adjust to equate the monopolist's bid to the aggregate net demand of the atomless part. Each trader belonging to the atomless part then obtains his Walrasian demand whereas monopolist's final holding is determined as the difference between his endowment and his bid, for the commodity he holds, and as the value of his bid in terms of relative prices, for the other commodity. They defined a monopoly equilibrium as a strategy played by the monopolist, corresponding to a positive bid of the commodity he holds, which guarantees him to obtain, via the trading process described above, a most preferred final holding among those he can achieve through his bids. They used their analytical framework to provide an economic theoretical foundation of the monopoly solution previously characterized in geometrical terms by Schydlowsky and Siamwalla (1966), under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and differentiable.[2]

In this paper, we consider the model of quantity-setting monopoly proposed by Busetto et al. (2023), under the simplifying assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible. This version of the model is appropriate for a comparison with the standard partial equilibrium analysis of monopoly.

Brown and Lee (2008), following Bork (1978), established a fundamental nexus between antitrust policy and allocative efficiency, interpreted as Pareto optimality. Nevertheless, their model did not allow them to establish an equivalent nexus among Pareto optimality, monopoly equilibrium, and the Walras equilibrium.

In this paper, we establish precisely such a nexus, under the assumption that the Walrasian demand of traders in the atomless part is invertible

[2]Moreover, they adapted to their monopoly bilateral exchange context the version of the Shapley window model used by Busetto et al. (2020) and they assumed that the atomless part behaves *à la* Cournot making bids of the commodity it holds. Then, they provided a sequential reformulation of the mixed version of the Shapley window model in terms of a two-stage game with observed actions where the quantity-setting monopolist moves first and the atomless part moves in the second stage, after observing the move of the monopolist in the first stage. This two-stage reformulation allowed them to provide a game theoretical foundation of the quantity-setting monopoly solution as they proved that the set of the allocations corresponding to a monopoly equilibrium and the set of those corresponding to a subgame perfect equilibrium of the two-stage game coincide.

and differentiable. Indeed, we prove a result which establishes an equivalence between the set of Pareto optimal monopoly allocations and the set of monopoly allocations, whenever the latter are also Walrasian. Moreover, we provide an example showing that this equivalence holds non-vacuously. These results are also extended to the core of the economy through a proposition showing an equivalence between the core and the set of monopoly allocations, whenever the latter are also Walrasian, and an example showing that this equivalence also holds non-vacuously.

Then, we use our model of a quantity-setting monopolist to confirm Aumann's argument, mentioned above, about the impossibility of a disadvantageous monopoly within the "classical theory." We prove two propositions establishing that monopoly is non-disadvantageous, that the price at a monopoly equilibrium is not lower than the price at a Walras equilibrium, and that the quantity supplied by the monopolist of the commodity he holds at a monopoly equilibrium is not greater than the quantity supplied at a Walras equilibrium.

Therefore, our Edgeworth framework supports the program of the refounding antitrust law on economic theory, in terms of allocative efficiency, proposed by Bork (1978). Moreover, it also reconciles this program with its reformulation in terms of consumer welfare, proposed by Lande (1982). Indeed, interpreting consumer welfare as the welfare of the atomless part, we provide a proposition showing that monopoly is non-advantageous, for each trader in the atomless part, with respect to the Walras equilibrium.

In recent years, a new approach to antitrust was proposed by a movement inspired by Judge Louis D. Brandeis and his strong antimonopoly credo, synthesized by the idea of the "curse of bigness" (see Brandeis (1914)). This movement, sometimes called the "New Brandeis School," can be seen as an alternative approach to antitrust, radically opposed to the Chicago School (see, for instance, Khan (2018), p. 131).

Our mixed Edgeworth box economy has a Brandeisian flavor in that – as stressed above – the monopolist, being an atom in a measure space, embodies the notion of "bigness." We prove a proposition showing that, when "bigness" is fully converted into market power and in no way resolves itself into price-taking, monopoly allocations are not Pareto optimal and they are advantageous for the monopolist and disadvantageous for each trader in the atomless part with respect to Walras allocations.

Therefore, in an Edgeworth box, both the Brandeis and the Chicago School turn out to be concerned with the "curse of bigness." Nevertheless, Khan (2018) claimed that the "curse of bigness" goes beyond market power by recalling: "Brandeis and many of his contemporaries feared that concentration of economic power aids the concentration of political power, and that such private power can itself undermine and overwhelm public government" (see p. 131).

Our Edgeworth box economy is too basic and abstract to represent the larger political consequences of "bigness," but it is rich enough to encompass some of its effects which exceed the rough exploitation of market power. In order to make this point, we reformulate and adapt to our Edgeworthian framework a result proved by Shitovitz (1997). His aim was to demonstrate "a class of monopolies where [...] the core yields a larger exploitation than at any Pareto optimal outcome that strictly dominates the monopolistic solution" (see p. 559-560).

Within our framework, we show that, when the monopolist exerts his market power without behaving as if he were a price-taker, for any monopoly allocation, there is an allocation in the core, which is neither a monopoly allocation nor a Walras allocation, and which is advantageous for the monopolist and non-advantageous for the atomless part with respect to that monopoly allocation.

The Shitovitz paradox shows that the "curse of bigness" may overwhelm the manifestation of atomic power in terms of monopolistic market power and provide a further advantage to the monopolist as argued by the New Brandeis School.

The paper is organized as follows. In Section 2, we introduce the mathematical model and we define the notion of a monopoly equilibrium. In Section 3, we assume that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and we derive its main properties. In Section 4, we analyze the basic general welfare properties of monopoly equilibrium. In Section 5, we analyze the consumer welfare properties of monopoly equilibrium. In Section 6, we introduce the Shitovitz paradox. In Section 7, we draw some conclusions and we suggest some further lines of research.

2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$.[3] Moreover, we assume that $T \setminus T_0 = \{m\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}^i(m) > 0$, $\mathbf{w}^j(m) = 0$ *and* $\mathbf{w}^i(t) = 0$, $\mathbf{w}^j(t) > 0$, *for each* $t \in T_0$, $i = 1$ *or* 2, $j = 1$ *or* 2, $i \neq j$.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^2_+ \to R$ is continuous, differentiable, strongly monotone, and strictly quasi-concave, for each $t \in T$, and $\frac{\partial u_t(x^i, 0))}{\partial x^j} = +\infty$, for each $x_i \in R_{++}$, whenever $\mathbf{w}^i(t) > 0$, for each $t \in T_0$.* [4]

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $D \times F$ such that $D \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

Assumption 4. *There is a coalition $\bar{T} \subseteq T_0$ such that $u_t(\cdot)$ is additively separable, i.e., $u_t(x) = v^i_t(x^i) + v^j_t(x^j)$, for each $x \in R^2_+$, $\frac{dv^i_t(x^i)}{dx^i} > 0$, $\frac{d^2 v^i_t((x^i)}{dx^{i2}} \leq 0$, for each $x^i \in R_+$, $\frac{dv^j_t(x^j)}{dx^j} > 0$, and $\frac{d^2 v^j_t(x^j)}{dx^{j2}} < 0$, for each $x^j \in R_+$, whenever $\mathbf{w}^i(t) > 0$, for each $t \in \bar{T}$.*

[3] The symbol 0 denotes the origin of R^2_+ as well as the real number zero: no confusion will result.

[4] Differentiability is to be understood as twice continuous differentiability and includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

We notice that all the propositions proved in Busetto et al. (2023) also hold under Assumptions 1, 2, and 3 as it is straightforward to verify that these assumptions imply their Assumptions 1, 2, 3, and 4. A price vector is a non-null vector $p \in R^2_+$. Let $\mathbf{X}^0 : T_0 \times R^2_{++} \to \mathcal{P}(R^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R^2_{++}$, $\mathbf{X}^0(t,p) = \text{argmax}\{u(x) : x \in R^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in R^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \{\int_{T_0} \mathbf{x}(t,p)\, d\mu : \mathbf{x}(\cdot,p)$ is integrable and $\mathbf{x}(t,p) \in \mathbf{X}^0(t,p)$, for each $t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t,\cdot)$ is nonempty and single-valued, by Assumption 2, it is possible to define the Walrasian demand of traders in the atomless part as the function $\mathbf{x}^0 : T_0 \times R^2_{++} \to R^2_+$ such that $\mathbf{X}^0(t,p) = \{\mathbf{x}^0(t,p)\}$, for each $t \in T_0$ and for each $p \in R^2_{++}$.

We reformulate now the following proposition, proved by Busetto et al. (2023).

Proposition 1. *Under Assumptions 1, 2, and 3, the function $\mathbf{x}^0(\cdot,p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for each $p \in R^2_{++}$.*

Proof. See the proof of Proposition 1 in Busetto et al. (2023). ■

A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector $p^* \in R^2_{++}$ and an allocation $\mathbf{x}^*$ such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$ and $u_t(\mathbf{x}^*(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : p^*x = p^*\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}^*$ for which there exists a price vector p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium.

Let $\mathbf{E}(m) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(m),\ i = 1,2\}$ denote the strategy set of atom m. We denote by $e \in \mathbf{E}(m)$ a strategy of atom m, where e_{ij}, $i,j = 1,2$, represents the amount of commodity i that atom m offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(m)$.

We then provide the following definition.

Definition 1. *Given a strategy $e \in \mathbf{E}(m)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^{2} e_{ij}\mu(m)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{2} e_{ji}\mu(m), \quad (1)$$

$j = 1,2$.

Market clearing price vectors can be normalized by Proposition 2 in Busetto et al. (2023). Henceforth, we say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R^2_+ : \sum_{i=1}^2 p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

We need to repropose now a proposition, proved by Busetto et al. (2023), which provides a necessary and sufficient condition for the existence of a market clearing price vector. In order to state it, we introduce the following preliminary definition.

Definition 2. *A square matrix C is said to be triangular if $c_{ij} = 0$ whenever $i > j$ or $c_{ij} = 0$ whenever $i < j$.*

Proposition 2. *Under Assumptions 1, 2, and 3, given a strategy $e \in \mathbf{E}(m)$, there exists a market clearing price vector $p \in \Delta \backslash \partial\Delta$ if and only if the matrix E is triangular.*

Proof. See the proof of Proposition 5 in Busetto et al. (2023). ■

We denote by $\pi(\cdot)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(m)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(\cdot)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(m)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(m)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}\mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta,\\ \mathbf{x}^j(m,e,p) &= \mathbf{w}^j(m), \text{ otherwise,}\end{aligned}$$

$j = 1, 2$,

$$\begin{aligned}\mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p), \text{ if } p \in \Delta \setminus \partial\Delta,\\ \mathbf{x}^j(t,p) &= \mathbf{w}^j(t), \text{ otherwise,}\end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a strategy $e \in \mathbf{E}(m)$ and a price selection $p(\cdot)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(m) = \mathbf{x}(m, e, p(e)),$$
$$\mathbf{x}(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$. Traders' final holdings constitute an allocation, by Proposition 6 in Busetto et al. (2023). Moreover, it is straightforward to verify that $p(e)\mathbf{x}(m, e, p(e)) = p(e)\mathbf{w}(m)$.

We can now provide the definition of a monopoly equilibrium.

Definition 3. *A strategy $\tilde{e} \in \mathbf{E}(m)$ such that $\tilde{E}$ is triangular is a monopoly equilibrium, with respect to a price selection $p(\cdot)$, if*

$$u_m(\mathbf{x}(m, \tilde{e}, p(\tilde{e})) \geq u_m(\mathbf{x}(m, e, p(e)),$$

for each $e \in \mathbf{E}(m)$.

A monopoly allocation is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium, with respect to a price selection $p(\cdot)$.

3 Monopoly equilibrium and invertible demand

As exposed in Busetto et al. (2023), the analysis of the monopoly problem in bilateral exchange can be simplified by introducing the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and compared, under this restriction, with the standard partial equilibrium analysis of monopoly.[5] In their Proposition 7, Busetto et al. (2023) proved that, when $\mathbf{w}^i(m) > 0$, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible if and only if, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$. Following those authors, we denote $p^{0i}(\cdot)$ denote the inverse of the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$. In their Proposition 8, Busetto et al. (2023) also proved that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, there exists a unique price selection $\mathring{p}(\cdot)$. By analogy with partial equilibrium analysis, $\mathring{p}(\cdot)$ can be interpreted as the inverse demand function of the monopolist. When the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the monopoly equilibrium can be reformulated as in Definition 3, with respect to monopolist's inverse demand function $\mathring{p}(\cdot)$.

[5] Theorem 1 in Cheng (1985) offers a set of sufficient conditions under which a system of demand functions satisfying gross substitutability can be inverted while the main theorem in Fisher (1972) provides a set of necessary and sufficient conditions on preferences which generates a system of demand functions satisfying gross substitutability. Recent work by Diasakos and Gerasimou (2022) provides a different set of necessary and sufficient conditions on preferences compatible with invertible demand. In this paper, our focus is on the welfare properties of the monopoly solution and the invertibility of the aggregate demand of the atomless part for the commodity held by the monopolist is assumed. We propose to develop in a future paper an analysis of the relationships between the studies on invertible demand functions mentioned above and our model of monopoly in order to obtain a specification of the preference conditions which guarantee invertibility in our context.

Now, given a price vector $(p^i, p^j) \in \Delta \setminus \partial\Delta$, with some abuse of notation, we denote by p both the scalar $p = \frac{p^i}{p^j}$ and the vector $(\frac{p^i}{p^j}, 1)$, whenever $\mathbf{w}^i(m) > 0$: it will be clear from the context when p denotes a scalar or a vector.

By means of the following proposition, we show that, when the aggregate demand of the atomless for the commodity held by the monopolist is invertible, the inverse demand function of the monopolist is strictly decreasing.

Proposition 3. *Under Assumptions 1, 2, and 3, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}. Then, the function $\mathring{p}(\cdot)$ is strictly decreasing on the set $\{e \in \mathbf{E}(m) : E$ is triangular $\}$.*

Proof. Let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible on R_{++}. The correspondence $\int_{T_0} \mathbf{X}^0(t, \cdot)\, d\mu$ is upper hemicontinuous at each $p \in R_{++}$, by the argument used in the proof of Property (ii) in Debreu (1982). But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu\}$ is continuous as $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$, for each $p \in R_{++}$, by Proposition 1. Moreover, the function $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$ is strictly monotone as it is continuous and invertible, by Theorem 4.4.2 in Dieudonné (1969). Then, the function $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu$ is strictly decreasing as it diverges to $+\infty$ when p converges to 0, by the argument used in the proof of Proposition 4 in Busetto et al. (2023). But then, the function $\mathring{p}(\cdot)$ is strictly decreasing on the set $\{e \in \mathbf{E}(m) : E$ is triangular $\}$ as $\mathring{p}(\cdot) = p^{0i}(\cdot)$ on this set and $p^{0i}(\cdot)$ is strictly decreasing on R_{++}, by Proposition 3.1.9 in Sohrab (2014). ■

By means of the following proposition, we then show that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and the Walrasian demand of traders in the atomless part is differentiable, the inverse demand function of the monopolist is differentiable with a strictly negative derivative.

Proposition 4. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Then, the function $\mathring{p}(\cdot)$ is differentiable and $\frac{d\mathring{p}(e)}{de_{ij}} < 0$, at each $e \in \mathbf{E}(m)$ such that E is triangular.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}, and the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$. We have that

$$p\mathbf{x}^{01}(t, p) + \mathbf{x}^{02}(t, p) = p\mathbf{w}^1(t) + \mathbf{w}^2(t),$$

for each $t \in T_0$ and for each $p \in R_{++}$, as $u_t(\cdot)$ is strongly monotone, for each $t \in T_0$, by Assumption 2. Differentiating with respect to p, we obtain

$$\mathbf{x}^{01}(t,p)dp + p\frac{d\mathbf{x}^{01}(t,p)}{dp}dp + \frac{d\mathbf{x}^{02}(t,p)}{dp}dp = \mathbf{w}^1(t)dp.$$

Then, we have that

$$\frac{d\mathbf{x}^{01}(t,p)}{dp} = \frac{\mathbf{w}^1(t) - \mathbf{x}^{01}(t,p) - \frac{d\mathbf{x}^{02}(t,p)}{dp}}{p} \leq \frac{\mathbf{w}^1(t)}{p}.$$

But then, the function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is differentiable on R_{++} and

$$\frac{d\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu}{dp} = \int_{T_0} \frac{d\mathbf{x}^{01}(t,p)}{dp}\, d\mu,$$

for each $p \in R_{++}$, as the function $\mathbf{x}^0(\cdot,p)$ is integrable, for each $p \in R_{++}$, by Proposition 1, and the function $\mathbf{x}^0(t,\cdot)$ is differentiable on R_{++}, for each $t \in T_0$, by Theorem 6.26 in Klenke (2020). We have that $\frac{d\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu}{dp} \leq 0$, for each $p \in R_{++}$, as $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ is strictly decreasing by the argument used in the proof of Proposition 3. Suppose that $\frac{d\int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu}{dp} = 0$ at some $\bar{p} \in R_{++}$. Then, it must be that $\frac{d\int_{\bar{T}} \mathbf{x}^{01}(t,\bar{p})\, d\mu}{dp} = 0$ as $\mu(\bar{T}) > 0$. Consider a trader $\tau \in T_0$. Then, it must be that $\frac{d\mathbf{x}^{01}(\tau,\bar{p})}{dp} = 0$. We have that $\mathbf{x}^{01}(\tau,\bar{p}) > 0$, by the argument used in the proof of Proposition 5 in Busetto et al. (2023). Suppose that $\mathbf{x}^{02}(\tau,\bar{p}) > 0$. For commodity 1, the substitution effect is negative and the income effect is nonpositive, by Assumption 4. Then, it must be that $\frac{d\mathbf{x}^{01}(\tau,\bar{p})}{dp} < 0$, by the Slutsky equation, a contradiction. Suppose that $\mathbf{x}^{02}(\tau,\bar{p}) = 0$. Then, we have that $\mathbf{x}^{01}(\tau,\bar{p}) = \frac{\mathbf{w}^2(\tau)}{\bar{p}}$. But then, we have that $\frac{d\mathbf{x}^{01}(\tau,\bar{p})}{dp} = -\frac{\mathbf{w}^2(\tau)}{\bar{p}^2} < 0$, a contradiction. Therefore, it must be that $\frac{d\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu}{dp} < 0$, for each $p \in R_{++}$. Hence, we have that $\frac{d\mathring{p}(e)}{de_{ij}} < 0$, at each $e \in \mathbf{E}(m)$, as $\frac{dp^{01}(x)}{dx} = (\frac{d\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu}{dp})^{-1} < 0$, by the inverse function theorem. ■

4 Allocative efficiency, Pareto optimality, and core: non-disadvantageous monopoly

Bork (1978) provocatively re-founded antitrust law on economic theory. In particular, he identified the main goal of antitrust with allocative efficiency. After a long standing debate on antitrust, Brown and Lee (2008), in a seminal paper in which analyzes antitrust issues within a general equilibrium framework, unambiguously interpreted allocative efficiency as Pareto optimality.

In the recasting of antitrust analysis based on the Edgeworth box model we propose in this paper, we follow this interpretation and we establish the optimality properties of the monopoly equilibrium introduced in the previous section.

In order to develop our analysis, we need to introduce the following further definitions. An allocation $\mathbf{x}$ is said to be individually rational if $u_t(\mathbf{x}(t)) \geq u_t(\mathbf{w}(t))$, for each $t \in T$. Moreover, an allocation $\mathbf{x}$ is said to be Pareto optimal if there is no allocation $\mathbf{y}$ such that $u_t(\mathbf{y}(t)) \geq u_t(\mathbf{x}(t))$, for each $t \in T$, and $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$, for a nonnull set of traders t in T. According to Shitovitz (1973), an efficiency equilibrium is defined as a pair $(\hat{p}, \hat{\mathbf{x}})$, where the price vector $\hat{p}$ and the allocation $\hat{\mathbf{x}}$ are such that $u_t(\hat{\mathbf{x}}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : \hat{p}x = \hat{p}\hat{\mathbf{x}}(t)\}$, for each $t \in T$.[6] Finally, an efficiency allocation is an allocation $\hat{\mathbf{x}}$ for which there exists a price vector $\hat{p} \in R^2_{++}$ such that the pair $(\hat{p}, \hat{\mathbf{x}})$ is an efficiency equilibrium.

Borrowing from Shitovitz (1973), we show now a proposition which establishes that a monopoly allocation is Pareto optimal if and only if it is an efficiency allocation. It provides a rationale for Pareto optimality as a criterion for allocative efficiency on the basis of the first and second fundamental theorems of welfare economics.

Proposition 5. *Under Assumptions 1, 2, and 3, let $\mathbf{w}^i(m) > 0$ and let $\tilde{\mathbf{x}}$ be monopoly allocation. Then, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if it is an efficiency allocation.*

Proof. Let $\mathbf{w}^i(m) > 0$ and let $\tilde{\mathbf{x}}$ be a monopoly allocation. Suppose that the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal. We adapt to our framework the argument used by Shitovitz (1973) to prove the corollary to his Lemma 1. It is straightforward to verify that $\tilde{\mathbf{x}}$ is individually rational.

[6] The notion of efficiency equilibrium coincides with that of equilibrium relative to a price system which was used by Debreu (1959) to prove the first and the second fundamental theorems of welfare economics.

Let $\tilde{\mathbf{G}} \to \mathcal{P}(R^2)$ be a correspondence such that $\tilde{\mathbf{G}}(t) = \{x - \tilde{\mathbf{x}}(t) : x \in R^2_+ \text{ and } u_t(x) > u_t(\tilde{\mathbf{x}}(t))\}$, for each $t \in T$. Moreover, let $\int_T \tilde{\mathbf{G}}(t)\, d\mu = \{\int_T \tilde{\mathbf{g}}(t)\, d\mu : \tilde{\mathbf{g}}(t)$ is integrable and $\tilde{\mathbf{g}}(t) \in \tilde{\mathbf{G}}(t)$, for each $t \in T\}$. The set $\{x \in R^2_+ : u_t(x) \geq u_t(\tilde{\mathbf{x}})\}$ is convex as $u_m(\cdot)$ is strictly quasi-concave, by Assumption 2. Then, it is straightforward to verify that the set $\tilde{\mathbf{G}}(m)$ is convex. But then, $\int_T \tilde{\mathbf{G}}(t)\, d\mu$ is convex, by Theorem 1 in Shitovitz (1973). We now prove that $0 \notin \int_T \tilde{\mathbf{G}}(t)\, d\mu$. Suppose that $0 \in \int_T \tilde{\mathbf{G}}(t)\, d\mu$. Then, there is an assignment $\mathbf{y}$ such that $u_t(\mathbf{y}(t)) > u_t(\tilde{\mathbf{x}}(t))$, for each $t \in T$, which is an allocation as $\int_T \mathbf{y}(t)\, d\mu = \int_T \tilde{\mathbf{x}}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. But then, $\tilde{\mathbf{x}}$ is not Pareto optimal, a contradiction. Therefore, it must be that $0 \notin \int_T \tilde{\mathbf{G}}(t)\, d\mu$. Then, there exists a vector $q \in R^2$ such that $(q \neq 0)$ and $q \int_T \tilde{\mathbf{G}}(t)\, d\mu \geq 0$, by the supporting hyperplane theorem. We know that $q \in R^2_{++}$, by the proof of Lemma 1 in Shitovitz (1973). Let $\hat{p} = \frac{q^i}{q^j}$. Then, the pair $(\hat{p}, \tilde{\mathbf{x}})$ is an efficiency equilibrium, by Lemma 1 in Shitovitz (1973). Therefore, the allocation $\tilde{\mathbf{x}}$ is an efficiency allocation. Conversely, suppose that the allocation $\tilde{\mathbf{x}}$ is an efficiency allocation. Then, the allocation $\tilde{\mathbf{x}}$ is Pareto optimal, by the first fundamental theorem of welfare economics. Hence, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if it is an efficiency allocation. ■

Brown and Lee (2008) observed: "[...] The nexus between Pareto optimality and antitrust law has been all but overlooked in the economic literature due to the singular focus on the deadweight loss analysis" (see p. 56). And they added: "Our analysis restores this nexus and suggests that the proper benchmark for measuring the cost of monopoly should be a Pareto optimal state of the economy, not simply competitive markets" (see pp. 56-57). Our Proposition 5 is linked to the work of these authors since it exhibits the nexus between Pareto optimality and antitrust in terms of allocative efficiency, in an Edgeworth box.

The next proposition shows a nexus between the Pareto-optimality properties of monopoly allocations and perfect competition. Indeed, it establishes that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and the Walrasian demand of each trader in the atomless part is differentiable, the set of Pareto optimal monopoly allocations and the set of monopoly allocations which are also Walrasian coincide.

Proposition 6. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a* *monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and $\mathbf{w}^2(m) = 0$. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Suppose that the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal. Moreover, suppose that $\tilde{\mathbf{x}}^2(t) = 0$, for each $t \in T_0$. Then, we have that

$$\tilde{\mathbf{x}}(m) = (\mathbf{w}^1(m) - \int_{T_0} \mathbf{x}^{01}(t, \mathring{p}(\tilde{e}))\, d\mu, \int_{T_0} \mathbf{w}^2(t)\, d\mu).$$

But then, we have that

$$\frac{du_m(\tilde{\mathbf{x}}(m))}{de} = -\frac{d\int_{T_0} \mathbf{x}^{01}(t, \mathring{p}(\tilde{e}))}{dp}\frac{d\mathring{p}(\tilde{e})}{de_{12}} < 0,$$

by Proposition 4, a contradiction. Therefore, there must be a coalition $O \subseteq T_0$ such $\tilde{\mathbf{x}}^2(t) \gg 0$, for each $t \in O$. There exists a vector $\hat{p} \in R^2_{++}$ such that the pair $(\hat{p}, \tilde{\mathbf{x}})$ is an efficiency equilibrium, by Proposition 5. Consider a trader $\tau \in O$. We have that

$$\frac{\frac{\partial u_\tau(\tilde{\mathbf{x}}(\tau))}{\partial x^1}}{\frac{\partial u_\tau(\tilde{\mathbf{x}}(\tau))}{\partial x^2}} = \tilde{p},$$

as $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e})) \gg 0$, for each $t \in O$. It must also be that

$$\frac{\frac{\partial u_\tau(\tilde{\mathbf{x}}(\tau))}{\partial x^1}}{\frac{\partial u_\tau(\tilde{\mathbf{x}}(\tau))}{\partial x^2}} = \hat{p},$$

as the pair $(\hat{p}, \hat{\mathbf{x}})$ is an efficiency equilibrium, for each $t \in O$. Then, we have that $\tilde{p} = \hat{p}$. But then, $\tilde{\mathbf{x}}$ is such that $\tilde{p}\tilde{\mathbf{x}}(t) = \tilde{p}\mathbf{w}(t)$ and $u_t(\tilde{\mathbf{x}}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(t)\}$, for each $t \in T$. Therefore, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Conversely, suppose that the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Then, it is straightforward to show that it is also an efficiency equilibrium. But then, the allocation $\tilde{\mathbf{x}}$ is Pareto optimal by the first fundamental theorem of welfare economics. Hence, the monopoly allocation $\tilde{\mathbf{x}}$ is Pareto optimal if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. ■

The next proposition provides a necessary and sufficient condition for a monopoly allocation to be Walrasian.

Proposition 7. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and $\mathbf{w}^2(m) = 0$. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Suppose that the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Moreover, suppose that $\tilde{\mathbf{x}}(m) \gg 0$. It must be that

$$\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}} = \tilde{p},$$

as the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Moreover, we have that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}(\tilde{p} + \frac{d\mathring{p}(\tilde{e})}{de_{12}}\tilde{e}_{12}) = 0,$$

as $\tilde{e}$ is a monopoly equilibrium. Then, we obtain that

$$\frac{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1}}{\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}} \neq \tilde{p},$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} < 0$, by Proposition 4, a contradiction. Therefore, it must be that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. Conversely, suppose that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. We have that $\tilde{\mathbf{x}}(m) = (0, \mathbf{w}^1(m)\tilde{p})$. Let $\tilde{x}^2(x^1)$ be a function such that $u_m(x^1, x^2(x^1)) = u_m(\tilde{\mathbf{x}}(m))$, for each $0 \leq x^1 \leq \mathbf{w}^1(m)$. We have that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}(\tilde{p} + \frac{d\mathring{p}(\tilde{e})}{de_{12}}\tilde{e}_{12}) \geq 0,$$

as $\tilde{e}_{12} = \mathbf{w}^1(m)$. Then, it must be that

$$-\frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^1} + \frac{\partial u_m(\tilde{\mathbf{x}}(m))}{\partial x^2}\tilde{p} > 0,$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} < 0$, by Proposition 4. But then, we have that $\frac{d\tilde{x}^2}{dx^1} > -\tilde{p}$, for each $0 \leq x^1 \leq \mathbf{w}^1(m)$, as $u_m(\cdot)$ is strictly quasi-concave, by Assumption 2.

Suppose that there exists a commodity bundle $\bar{x} \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(m)\}$ such that $u_m(\bar{x}) > u_m(\tilde{\mathbf{x}}(m))$. Then, it must be that $\bar{x}^2 > \tilde{x}^2(\bar{x}^1)$ as $u_m(\cdot)$ is strongly monotone, by Assumption 2. But then, by the mean value theorem, there exists some x' such that $0 < x' < \bar{x}^1$ and such that

$$\frac{d\tilde{x}^2(x'^1)}{dx^1} = \frac{\tilde{x}^2(\bar{x}^1) - \tilde{x}^2(0)}{\bar{x}^1 - 0} < -\tilde{p},$$

a contradiction. Therefore, we have that $u_m(\tilde{\mathbf{x}}(m) \geq u_m(y)$, for all $y \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(m)\}$. Hence, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$. ■

The following proposition is an immediate consequence of Proposition 7: under the same assumptions, it provides a characterization of Pareto optimal monopoly allocations.

Proposition 8. *Under Assumptions 1, 2, 3, and 4, let* $\mathbf{w}^i(m) > 0$ *and let the function* $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ *be invertible on* R_{++} *and the function* $\mathbf{x}^0(t,\cdot)$ *be differentiable on* R_{++}*, for each* $t \in T_0$*. Moreover, let* $\tilde{e} \in \mathbf{E}(m)$ *be a monopoly equilibrium, with respect to the unique price selection* $\mathring{p}(\cdot)$*, and let* $\tilde{p} = \mathring{p}(\tilde{e})$*,* $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$*,* $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$*, for each* $t \in T_0$*. Then,* $\tilde{\mathbf{x}}$ *is Pareto optimal if and only if* $\tilde{e}_{ij} = \mathbf{w}^i(m)$*.*

Proof. Let $\mathbf{w}^i(m) > 0$ and let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Suppose that $\tilde{\mathbf{x}}$ is Pareto optimal. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 6. But then, we have that $\tilde{e}_{ij} = \mathbf{w}^i(m)$, by Proposition 7. Conversely, suppose that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 7. But then, $\tilde{\mathbf{x}}$ is Pareto optimal, by Proposition 6. Hence, $\tilde{\mathbf{x}}$ is Pareto optimal if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$. ■

We provide now an example showing that Propositions 6, 7, and 8 hold non-vacuously.

Example 1. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4.* $T_0 = [0,1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \frac{1}{8}x^1 + \sqrt{x^2}$, T_0 *is taken with Lebesgue measure,* $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x^1} + x^2$*, for each* $t \in T_0$*. Then, there is a unique monopoly allocation* $\tilde{\mathbf{x}}$ *which coincides with the unique Walras allocation* $\mathbf{x}^*$*.*

Proof. The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$, where $\tilde{e}_{12} = 1$, and the allocation $\tilde{\mathbf{x}}$ such that $(\tilde{\mathbf{x}}^1(m), \tilde{\mathbf{x}}^2(m)) = (0, \frac{1}{2})$ and $(\tilde{\mathbf{x}}^1(t),$

$\tilde{\mathbf{x}}^2(t)) = (1, \frac{1}{2})$, for each $t \in T_0$, is the unique monopoly allocation. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $p^* = \frac{1}{2}$, and the allocation $\mathbf{x}^*$ is such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (0, \frac{1}{2})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (1, \frac{1}{2})$, for each $t \in T_0$. Hence, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ which coincides with the unique Walras allocation $\mathbf{x}^*$. ∎

We consider now the relationship between the set of monopoly allocations and the set of allocations belonging to the core. The core can be seen as a criterion of allocative efficiency stricter than Pareto optimality as is well known that any allocation in the core is Pareto optimal whereas the converse does not necessarily hold.

We say that an allocation $\mathbf{y}$ dominates an allocation $\mathbf{x}$ via a coalition S if $u_t(\mathbf{y}(t)) \geq u_t(\mathbf{x}(t))$, for each $t \in S$, $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$ for a non-null subset of traders t in S, and $\int_S \mathbf{y}(t)\, d\mu = \int_S \mathbf{w}(t)\, d\mu$. The core is the set of all allocations which are not dominated via any coalition.

The following proposition is a straightforward consequence of Proposition 6: under the same assumptions, it establishes an equivalence between the core and the set of monopoly allocations, whenever the latter are also Walrasian.

Proposition 9. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, the monopoly allocation $\tilde{\mathbf{x}}$ is in the core if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium.*

Proof. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to a price selection $p(\cdot)$. Suppose that the monopoly allocation $\tilde{\mathbf{x}}$ is in the core. Then, $\tilde{\mathbf{x}}$ is Pareto optimal. But then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 6. Conversely, suppose that the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. Then, the allocation $\tilde{\mathbf{x}}$ is in the core, by the same argument used by Aumann (1964) in the proof of his main theorem. Hence, the allocation $\tilde{\mathbf{x}}$ is in the core if and only if the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium. ∎

The next proposition is an immediate consequence of Proposition 8: under the same assumptions, it provides a characterization of monopoly allocations which are in the core.

Proposition 10. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++} and the function $\mathbf{x}^0(t, \cdot)$*

be differentiable on R_{++}, for each $t \in T_0$. Moreover, let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Then, $\tilde{\mathbf{x}}$ is in the core if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$.

Proof. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the price selection $\mathring{p}(\cdot)$. Suppose that $\tilde{\mathbf{x}}$ is in the core. Then, $\tilde{\mathbf{x}}$ is Pareto optimal. But then, we have that $\tilde{e}_{ij} = \mathbf{w}^i(m)$, by Proposition 8. Conversely, suppose that $\tilde{e}_{ij} = \mathbf{w}^i(m)$. Then, the pair $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium, by Proposition 7. But then, $\tilde{\mathbf{x}}$ is in the core, by the same argument used by Aumann (1964) in the proof of his main theorem. Hence, $\tilde{\mathbf{x}}$ is in the core if and only if $\tilde{e}_{ij} = \mathbf{w}^i(m)$. ■

Example 1 shows that Propositions 6, 7, and 8 hold non-vacuously. Moreover, for the same exchange economy, we can now show that the core does not coincide with the set of Walras equilibria.

Example 1′. *Consider the exchange economy specified in Example 1. Then, the core does not coincide with the set of Walras equilibria.*

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $p^* = \frac{1}{2}$ and the allocation $\mathbf{x}^*$ is such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (0, \frac{1}{2})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t))$ $= (1, \frac{1}{2})$, for each $t \in T_0$. The core consists of all the allocations $\mathbf{x}$ of the form $(\mathbf{x}^1(m), \mathbf{x}^2(m)) = (0, 1-\alpha)$ and $(\mathbf{x}^1(t), \mathbf{x}^2(t)) = (1, \alpha)$, for each $t \in T_0$, where $0 \leq \alpha \leq \frac{1}{2}$, as for such allocations the pair $(p^*, \mathbf{x})$ is an efficiency equilibrium and $p^*\mathbf{x}(t) \leq p^*\mathbf{w}(t)$, for each $t \in T_0$, by Theorem A^* in Shitovitz (1973). Hence, the core does not coincide with the set of Walras equilibria. ■

In our Example 1′ – as in Example 1 in Shitovitz (1973) – the unique Walras allocation is worse, in terms of the monopolists utility, than any other allocation in the core. Shitovitz (1973), at the end of a discussion of his result, stressed this point as an open problem, which was in turn reproposed by Aumann (1973) through the following conjecture: "In a monopolistic market, for each core allocation $\mathbf{x}$ there is a competitive allocation $\mathbf{y}$ whose utility to the monopolist is $\leq$ that of $\mathbf{x}$" (see p. 1). Nonetheless, this author provided three examples, in the same bilateral monopolistic framework of Shitovitz' Example 1, which invalidate his initial conjecture since they show that monopoly may be, according to his terminology, "disadvantageous." As already reminded, Aumann (1973) first of all stressed the relevance of the double characterization of the monopolist both as an atom and as an agent who initially holds a corner on one of the two commodities (see p. 2). Then, in the discussion of his counterintuitive examples he suggested: "Perhaps what is needed at this stage is a careful reappraisal of the ideas underlying the use of core in economic analysis" (see p. 9). Moreover, in considering his results from the point of view of the classical economic theory, he affirmed: "The kind of phenomenon illustrated for the core [...] is of course impossible in classical theory. If the monopolist sets prices, he cannot end up worse off that at the competitive equilibrium, since he always has the option of setting the prices equal to competitive prices" (see p. 9). It is at this point that, in the passage quoted in the Introduction, he affirmed the need of a theory that is applicable in any market, and when applied to a monopoly, yields the price-taking mechanism (see p. 10). Then he added: "To put the argument differently, one feels on an intuitive, common sense level that the monopolist has a distinct advantage; but economic theory, rather than explaining this phenomenon, simply states it in a specific form. For an explanation, one looks to game theory; but evidently, the game-theoretic notion of core is not the proper vehicle for such an explanation" (see p. 10). However, Aumann (1973) did not develop a full theory of a price-setting monopolist in bilateral exchange.

Borrowing from Busetto et al. (2020), Busetto et al. (2023) provided a sequential reformulation of the mixed version of the Shapley window model for the same bilateral exchange economy considered in Section 2 above. Their sequential structure was expressed as a two-stage game with observed actions where the quantity-setting monopolist m – an atom with a "corner" of one of the commodities like in Aumann (1973) – moves first and the atomless part moves in the second stage, after observing the moves of the monopolist in the first stage. This two-stage structure allowed the authors to provide a game theoretical foundation of the quantity-setting monopoly solution: indeed they proved that the set of the allocations corresponding to a monopoly equilibrium and the set of those corresponding to a subgame perfect equilibrium of the two-stage game coincide.

The game theoretical foundation of monopoly equilibrium proposed by Busetto et al. (2023) allows each trader to behave strategically and endogenously generates "the price-taking mechanism" concerning the atomless part, as wished by Aumann (1973) in the passage mentioned above, while consolidating the structural monopolistic power of the atom as a first mover in the two-stage game.

We now use our model of a quantity-setting monopolist to confirm the other argument raised by Aumann (1973), concerning the impossibility of a disadvantageous monopoly within the "classical theory": the following two propositions show, respectively, that monopoly is non-disadvantageous, that the price at a monopoly equilibrium is not inferior to the price at a Walras equilibrium, and that the quantity supplied by the monopolist of the commodity he holds at a monopoly equilibrium is not superior to the quantity supplied at a Walras equilibrium.

Proposition 11. *Under Assumptions 1, 2, and 3, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}. If $\tilde{\mathbf{x}}$ is a monopoly allocation and $\mathbf{x}^*$ is a Walras allocation, then $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a price p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. We have that

$$\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu = \int_{T_0} \mathbf{x}^{*1}(t)\, d\mu = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m),$$

as $\mathbf{x}^*$ is a Walras allocation. Suppose that $\mathbf{x}^{*1}(m) = \mathbf{w}^1(m)$. Then, we have that $\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu = 0$. But then, we have that $\mathbf{x}^{01}(t, p^*) = 0$, for each $t \in T_0$. Consider a trader $\tau \in T_0$. It must be that $\mathbf{x}^{02}(\tau, p^*) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. Then, we have that $\frac{\partial u_\tau(\mathbf{x}^0(\tau, p^*))}{\partial x^1} = +\infty$, by Assumption 2, and $\frac{\partial u_\tau(\mathbf{x}^0(\tau, p^*))}{\partial x^1} \leq \lambda p^*$ and $\frac{\partial u_\tau(\mathbf{x}^0(\tau, p^*))}{\partial x^2} = \lambda$, by the necessary conditions of the Kuhn-Tucker theorem. But then, it must be that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is differentiable. Therefore, we have that $\mathbf{x}^{*1}(m) < \mathbf{w}^1(m)$. Let $e^* \in \mathbf{E}(m)$ be a strategy such that $e^*_{12} = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m)$. We have that

$$\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu = \int_{T_0} \mathbf{x}^{*1}(t)\, d\mu = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m) = e^*_{12} = \int_{T_0} \mathbf{x}^{01}(t, \mathring{p}(e^*)),$$

as the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium and the function $\mathring{p}(\cdot)$ is the unique price selection. Then, it must be that $p^* = \mathring{p}(e^*)$. We have that

$$\mathbf{x}^{*1}(m) = \mathbf{w}^1(m) - e^*_{12} = \mathbf{x}^1(m, e^*, \mathring{p}(e^*))$$

and

$$\mathbf{x}^{*2}(m) = p^*\mathbf{w}^1(m) - p^*\mathbf{x}^{*1}(m) = p^* e^*_{12} = \mathbf{x}^2(m, e^*, \mathring{p}(e^*)).$$

Then, we have that

$$u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))) = u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m)) = u_m(\mathbf{x}(m, e^*, \mathring{p}(e^*)),$$

as the strategy $\tilde{e} \in \mathbf{E}(m)$ is a monopoly equilibrium. Hence, if $\tilde{\mathbf{x}}$ is a monopoly allocation and $\mathbf{x}^*$ is a Walras allocation, then $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$. ∎

Proposition 12. *Under Assumptions 1, 2, and 3, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}. If $\tilde{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium, then there exists a strategy $e^* \in \mathbf{E}(m)$ such that $p^* = \mathring{p}(e^*)$, $\tilde{p} \geq p^*$, where $\tilde{p} = \mathring{p}(\tilde{e})$, and $\tilde{e}_{ij} \leq e^*_{ij}$*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Moreover, let the pair $(p^*, \mathbf{x}^*)$ be a Walras equilibrium. There exists a strategy $e^* \in \mathbf{E}(m)$ such that $e^*_{12} = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m)$ and such that $p^* = \mathring{p}(e^*)$, by the same argument used in the proof of Proposition 11. Suppose that $\tilde{p} < p^*$. Consider a trader $\tau \in T_0$. Suppose that $\mathbf{x}^*(\tau) = (0, \mathbf{w}^2(\tau))$. Then, it must be that $\tilde{\mathbf{x}}(\tau) = (0, \mathbf{w}^2(\tau))$ as $\tilde{p} < p^*$. But then, we have that $u_\tau(\tilde{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}^*(\tau))$. Suppose that $\mathbf{x}^*(\tau) \neq (0, \mathbf{w}^2(\tau))$. Then, it must be that $\tilde{p}\mathbf{x}^*(\tau) < \mathbf{w}^2(\tau)$. But then, there exists a commodity bundle x' such that $\tilde{p}x' = \mathbf{w}^2(\tau)$ and $u_\tau(x') > u_\tau(\mathbf{x}^*(\tau))$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. Thus, we have that $u_\tau(\tilde{\mathbf{x}}(\tau)) > u_\tau(\mathbf{x}^*(\tau))$ as $u_\tau(\tilde{\mathbf{x}}(\tau)) \geq u_\tau(x')$. Therefore, we have that $u_t(\tilde{\mathbf{x}}(t)) \geq u_t(\mathbf{x}^*(t))$, for each $t \in T_0$. Moreover, we have that $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$, by Proposition 11. It must be that $\int_{T_0} \mathbf{x}^{01}(t, p^*)\, d\mu > 0$, by the same argument used in the proof of Proposition 11. Consider a trader $\tau \in T_0$. Then, we have that $\mathbf{x}^*(\tau) \neq (0, \mathbf{w}^2(\tau))$. But then, it must be that $u_\tau(\tilde{\mathbf{x}}(\tau)) > u_\tau(\mathbf{x}^*(\tau))$, by the previous argument. Therefore, the Walras allocation $\mathbf{x}^*$ is not Pareto optimal as that $u_t(\tilde{\mathbf{x}}(t)) > u_t(\mathbf{x}^*(t))$, for each $t \in T_0$, a contradiction. Hence, we have that $\tilde{p} \geq p^*$ and $\tilde{e}_{ij} \leq e^*_{ij}$ as the function $\mathring{p}(\cdot)$ is strictly decreasing on the set $\{e \in \mathbf{E}(m) : E \text{ is triangular }\}$, by Proposition 3. ∎

5 Consumer welfare and atomless part welfare: advantageous monopoly

In his celebrated paper, Lande (1982) criticized the path-breaking approach proposed by Bork for antitrust, based on allocative efficiency. He suggested that consumer welfare instead of allocative efficiency should be the goal promoted by the Sherman Act. Within our analysis of monopoly in the Edgheworth box, consumer welfare can be interpreted as the welfare of the atomless part of the economy.

The next proposition shows that monopoly is non-advantageous compared with "perfect competition," for each trader in the atomless part.

Proposition 13. *Under Assumptions 1, 2, and 3, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}. If $\tilde{\mathbf{x}}$ is a monopoly allocation and $\mathbf{x}^*$ is a Walras allocation, then $u_t(\tilde{\mathbf{x}}(t)) \leq u_t(\mathbf{x}^*(t))$, for each $t \in T_0$.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a price p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that $\tilde{p} \geq p^*$, by Proposition 12. Consider a trader $\tau \in T_0$. Consider the case where $\tilde{p} = p^*$. Then, it must be that $u_\tau(\tilde{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}^*(\tau))$. Consider the case where $\tilde{p} > p^*$. Suppose that $\tilde{\mathbf{x}}(\tau) = (0, \mathbf{w}^2(\tau))$. Then, it must be that $\mathbf{x}^*(\tau) = (0, \mathbf{w}^2(\tau))$ as $\tilde{p} > p^*$. But then, we have that $u_\tau(\tilde{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}^*(\tau))$. Suppose that $\tilde{\mathbf{x}}(\tau) \neq (0, \mathbf{w}^2(\tau))$. Then, it must be that $p^*\tilde{\mathbf{x}}(\tau) < \mathbf{w}^2(\tau)$. But then, there exists a commodity bundle x' such that $p^*x' = \mathbf{w}^2(\tau)$ and $u_\tau(x') > u_\tau(\tilde{\mathbf{x}}(\tau))$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2. Thus, we have that $u_\tau(\tilde{\mathbf{x}}(\tau)) < u_\tau(\mathbf{x}^*(\tau))$ as $u_\tau(\mathbf{x}^*(\tau)) \geq u_\tau(x')$. Hence, we have that $u_t(\tilde{\mathbf{x}}(t)) \leq u_t(\mathbf{x}^*(t))$, for each $t \in T_0$. ■

Example 1 in Section 4 exhibits a case of non-disadvantageous monopoly as the unique monopoly allocation coincides with the unique Walras allocation. In contrast, the next example exhibits the case of an advantageous monopoly as the monopolist strictly prefers his assignment at the unique monopoly allocation to that at the unique Walras allocation.

Example 2. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4. $T_0 = [0, 1]$, $T \setminus T_0 = \{m\}$, $\mu(m) = 1$, $\mathbf{w}(m) = (1, 0)$, $u_m(x) = \frac{1}{2}x^1 + \sqrt{x^2}$, T_0 is taken with Lebesgue measure, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x^1} + x^2$, for each $t \in T_0$. Then, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ and a unique Walras allocation $\mathbf{x}^*$ such that $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$.*

Proof. The unique monopoly equilibrium is the strategy $\tilde{e} \in \mathbf{E}(m)$, where $\tilde{e}_{12} = \frac{1}{4}$, and the allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$ and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$, is the unique monopoly allocation. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $p^* = (\frac{1}{4})^{\frac{1}{3}}$, and the allocation $\mathbf{x}^*$ such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (1 - (\frac{1}{4})^{\frac{1}{3}}, (\frac{1}{4})^{\frac{2}{3}})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = ((\frac{1}{4})^{\frac{1}{3}}, 1 - (\frac{1}{4})^{\frac{2}{3}})$, for each $t \in T_0$, is the unique Walras allocation. Moreover, we have that

$$u_m(\tilde{\mathbf{x}}(m)) = u_m\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{7}{8} > \frac{1}{2}+\frac{1}{2}(\frac{1}{4})^{\frac{1}{3}} = u_m\left(1 - (\frac{1}{4})^{\frac{1}{3}}, (\frac{1}{4})^{\frac{2}{3}}\right) = u_m(\mathbf{x}^*).$$

Hence, there is a unique monopoly allocation $\tilde{\mathbf{x}}$ and a unique Walras allocation $\mathbf{x}^*$ such that $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$. ■

The following result is a corollary to Proposition 11 establishing that monopoly is advantageous with respect to all Walras allocations which are not monopoly allocations.

Corollary 1. *Under Assumptions 1, 2, and 3, let $\mathbf{w}^i(m) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}. If $\tilde{\mathbf{x}}$ is a monopoly allocation, then $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$, for each Walras allocation $\mathbf{x}^*$ which is not a monopoly allocation.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation which is not a monopoly allocation. Then, it must be that $\tilde{\mathbf{x}} \neq \mathbf{x}^*$. There exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a relative price p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium. Moreover, there exists a strategy e^* such that $e^*_{12} = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m)$ and such that $p^* = \mathring{p}(e^*)$, by the same argument used in the proof of Proposition 11. We have that

$$\mathbf{x}^1(m, e^*, \mathring{p}(e^*)) = \mathbf{w}^1(m) - e^*_{12} = \mathbf{x}^{*1}(m)$$

and

$$\mathbf{x}^2(m, e^*, \mathring{p}(e^*)) = p^* e^*_{12} = p^*\mathbf{w}^1(m) - p^*\mathbf{x}^{*1}(m) = \mathbf{x}^{*2}(m).$$

Suppose that $u_m(\tilde{\mathbf{x}}(m)) = u_m(\mathbf{x}^*(m)$. Then, we have that

$$\begin{aligned} &u_m(\mathbf{x}(m, e^*, \mathring{p}(e^*))) = u_m(\mathbf{x}^*(m)) = u_m(\tilde{\mathbf{x}}(m)) = u_m(\mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))) \\ &\geq u_m(\mathbf{x}(m, e, \mathring{p}(e))), \end{aligned}$$

for each $e \in \mathbf{E}(m)$, as the strategy $\tilde{e}$ is a monopoly equilibrium. But then, the strategy e^* is a monopoly equilibrium, a contradiction. Hence, it must be that $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$, as $u_m(\tilde{\mathbf{x}}(m)) \geq u_m(\mathbf{x}^*(m))$, by Proposition 11. ■

The next result – a corollary to Proposition 12 – shows that, when monopoly is advantageous, the price at a monopoly equilibrium is greater than the price at any non-monopolistic Walras equilibrium and the quantity supplied by the monopolist of the commodity he holds at a monopoly equilibrium is lower than the quantity supplied at any non-monopolistic Walras equilibrium.

Corollary 2. *Under Assumptions 1, 2, and 3, let* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *be invertible on* R_{++}*. If* $\tilde{e} \in \mathbf{E}(m)$ *is a monopoly equilibrium, with respect to the unique price selection* $\mathring{p}(\cdot)$*, and the pair* $(p^*, \mathbf{x}^*)$ *is a Walras equilibrium such that* $\mathbf{x}^*$ *is not a monopoly allocation, then there exists a strategy* $e^* \in \mathbf{E}(m)$ *such that* $p^* = \mathring{p}(e^*)$*,* $\tilde{p} > p^*$*, where* $\tilde{p} = \mathring{p}(\tilde{e})$*, and* $\tilde{e}_{ij} < e^*_{ij}$

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\tilde{e} \in \mathbf{E}(m)$ be a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and let $\tilde{p} = \mathring{p}(\tilde{e})$, $\tilde{\mathbf{x}}(m) = \mathbf{x}(m, \tilde{e}, \mathring{p}(\tilde{e}))$, and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\tilde{e}))$, for each $t \in T_0$. Moreover, let the pair $(p^*, \mathbf{x}^*)$ be a Walras equilibrium such that $\mathbf{x}^*$ is not a monopoly allocation. Then, it must be that $\tilde{\mathbf{x}} \neq \mathbf{x}^*$. There exists a strategy e^* such that $e^*_{12} = \mathbf{w}^1(m) - \mathbf{x}^{*1}(m)$ and such that $p^* = \mathring{p}(e^*)$, by the same argument used in the proof of Proposition 11. It must be that $\tilde{e}_{ij} \neq e^*_{ij}$ as $\tilde{\mathbf{x}}(m) \neq \mathbf{x}^*(m)$. Then, we have that $\tilde{p} = \mathring{p}(\tilde{e}) \neq \mathring{p}(e^*) = p^*$ as the function $\mathring{p}(\cdot)$ is strictly decreasing on the set $\{e \in \mathbf{E}(m) : E \text{ is triangular }\}$, by Proposition 3. Hence, we have that $\tilde{p} > p^*$ as $\tilde{p} \geq p^*$, by Proposition 12, and $\tilde{e}_{ij} < e^*_{ij}$ as the function $\mathring{p}(\cdot)$ is strictly decreasing on the set $\{e \in \mathbf{E}(m) : E \text{ is triangular }\}$, by Proposition 3. ∎

The next corollary to Proposition 13 shows that monopoly is disadvantageous with respect to all Walras allocations which are not monopoly allocations, for each trader in the atomless part.

Corollary 3. *Under Assumptions 1, 2, and 3, let* $\mathbf{w}^i(m) > 0$ *and let the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *be invertible on* R_{++}*. If* $\tilde{\mathbf{x}}$ *is a monopoly allocation, then* $u_t(\tilde{\mathbf{x}}(t)) < u_t(\mathbf{x}^*(t))$*, for each* $t \in T_0$*, and for each Walras allocation* $\mathbf{x}^*$ *which is not a monopoly allocation.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and let $\mathbf{x}^*$ be a Walras allocation which is not a monopoly allocation. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$, and a price p^* such that the pair

$(p^*, \mathbf{x}^*)$ is a Walras equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that $\tilde{p} > p^*$, by Corollary 2. Hence, we have that $u_t(\tilde{\mathbf{x}}(t)) < u_t(\mathbf{x}^*(t))$, for each $t \in T_0$, by the same argument used in the proof of Proposition 13. ■

Recently, a new school in antitrust, named "New Brandeis School," has emerged. According to Khan (2018) this school "signals a break with the Chicago School, whose ideas set antitrust on a radically new course starting in the 1970s and 1980s and continue to underpin competition policy in the USA today" (see p. 131). The new school takes inspiration from the ideas of Judge Louis D. Brandeis whose antimonopoly attitude was expressed by the famed credo about the "curse of bigness" (see Brandeis (1914)). Our model incorporates the Brandeisian notion of "bigness" in that the monopolist m is an atom. As we have previously reminded, the two-stage foundation of the quantity-setting monopoly model provided by Busetto et al. (2023) captures this structural feature conferring to the monopolist the prerogative to convert his "bigness" into market power through his strategic behavior.

We can now show that, when the monopolist, in exerting his power, does not behave as if he were a price-taker, monopoly allocations are not Pareto optimal and they are advantageous for the monopolist and disadvantageous for each trader in the atomless part compared with Walras allocations.

Proposition 14. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}, and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. If the set of monopoly allocations and the set of Walras allocations are disjoint, $\tilde{\mathbf{x}}$ is a monopoly allocation, and $\mathbf{x}^*$ is a Walras allocation, then $\tilde{\mathbf{x}}$ is not Pareto optimal, $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$, and $u_t(\tilde{\mathbf{x}}(t)) < u_t(\mathbf{x}^*(t))$, for each $t \in T_0$.*

Proof. Assume that $\mathbf{w}^i(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible on R_{++}, and the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$. Suppose that the set of monopoly allocations and the set of Walras allocations are disjoint. Let $\tilde{\mathbf{x}}$ be a monopoly allocation and $\mathbf{x}^*$ be a Walras allocation. Hence, $\tilde{\mathbf{x}}$ is not Pareto optimal as it is not a Walras allocation, by Proposition 6, $u_m(\tilde{\mathbf{x}}(m)) > u_m(\mathbf{x}^*(m))$, by Corollary 1, and $u_t(\tilde{\mathbf{x}}(t)) < u_t(\mathbf{x}^*(t))$, for each $t \in T_0$, by Corollary 3. ■

6 The Shitovitz paradox: advantageous monopoly in the core

Proposition 14 reconciles, in the abstract and terse framework of the Edgeworth box, the Chicago and the New Brandeis Schools: when monopoly power never leads to a price-taking behavior by the monopolist, the "curse of bigness" undermines allocative efficiency – i.e., Pareto optimality – and consumer welfare – i.e., the welfare of the atomless part of the economy. Clearly, the institutional parsimoniousness of an Edgeworth box economy does not permit one to capture the broader influence of "bigness" on polity and society but it is rich enough to allow for an insight into its main economic features, beyond the mere exercise of market power in terms of strategic quantity-setting. Indeed, by adapting to our framework an argument proposed by Shitovitz (1997), we shall exhibit a new kind of antitrust paradox: when the monopolist, in exerting his power, does not behave as if he were a price-taker, for any monopoly allocation there is an allocation in the core, which is neither a monopoly allocation nor a Walras allocation, and which is advantageous for the monopolist and non-advantageous for the atomless part compared with that monopoly allocation. This version of Shitovitz' paradox highlights that the "curse of bigness" is not exhausted by the strategic exploitation of monopoly power but that it can manifest itself in other ways, as predicted by the New Brandeis School.

In order to develop the Shitovitz argument in our bilateral exchange framework, we need to introduce the following definition: we say that an allocation $\mathbf{y}$ weakly dominates an allocation $\mathbf{x}$ via a coalition S if $u_t(\mathbf{y}(t)) > u_t(\mathbf{x}(t))$, for each $t \in S$, and $\int_S \mathbf{y}(t)\, d\mu = \int_S \mathbf{w}(t)\, d\mu$. The weak core is the set of all allocations which are not weakly dominated via any coalition.

Shitovitz (1997) considered a maximization problem, which can be adapted to our monopoly model and which will be henceforth referred to as the Shitovitz maximization problem. In our case, it can be stated as follows

$$\begin{aligned}
&\max_{\mathbf{x}} u_m(\mathbf{x}(m))\\
&\text{subject to}\\
&u_t(\mathbf{x}(t)) \geq u_t(\mathbf{w}(t)), \text{ for each } t \in T_0,\\
&\int_T \mathbf{x}(t)\, d\mu \leq \int_T \mathbf{w}(t)\, d\mu.
\end{aligned}$$

The following proposition was proved by Shitovitz (1997).

Proposition 15. *Under Assumptions 1, 2, and 3, there exists a solution* $\check{\mathbf{x}}$ *to the Shitovitz maximization problem.*

Proof. See the proof of Proposition 2.1 in Shitovitz (1997). ■

We prove now two corollaries to Proposition 15, which provide a characterization of the solutions to the Shitovitz maximization problem.

Corollary 4. *Under Assumptions 1, 2, and 3, if* $\check{\mathbf{x}}$ *is a solution to the Shitovitz maximization problem, then* $\int_T \check{\mathbf{x}}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$.

Proof. Let $\check{\mathbf{x}}$ be a solution to the Shitovitz maximization problem. Then, we have that $\check{\mathbf{x}}(m)\mu(m) \leq \int_T \mathbf{w}(t)\, d\mu - \int_{T_0} \check{\mathbf{x}}(t)\, d\mu$. Suppose that $\mathbf{x}(m)\mu(m) \neq \int_T \mathbf{w}(t)\, d\mu - \int_{T_0} \check{\mathbf{x}}(t)\, d\mu$. Then, there is a vector $\bar{x}$ such that $\check{\mathbf{x}}(m)\mu(m) \leq \bar{x}\mu(m) \leq \int_T \mathbf{w}(t)\, d\mu - \int_{T_0} \check{\mathbf{x}}(t)\, d\mu$ and $\check{\mathbf{x}}(m) \neq \bar{x}$. Let $\bar{\mathbf{x}}$ be an assignment such that $\bar{\mathbf{x}}(m) = \bar{x}$ and $\bar{\mathbf{x}}(t) = \check{\mathbf{x}}(t)$, for each $t \in T_0$. Then, $\bar{\mathbf{x}}$ satisfies the constraints of the Shitovitz maximization problem and $u_m(\bar{\mathbf{x}}(m)) > u_m(\check{\mathbf{x}}(m))$ as $u_m(\cdot)$ is strongly monotone, by Assumption 2, a contradiction. Hence, it must be that $\int_T \check{\mathbf{x}}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. ■

Corollary 5. *Under Assumptions 1, 2, and 3, if* $\check{\mathbf{x}}$ *is a solution to the Shitovitz maximization problem, then* $u_t(\check{\mathbf{x}}(t)) = u_t(\mathbf{w}(t))$, *for each* $t \in T_0$.

Proof. Let $\check{\mathbf{x}}$ be a solution to the Shitovitz maximization problem. Suppose that $u_t(\check{\mathbf{x}}(t)) > u_t(\mathbf{w}(t))$, for each $t \in R$, where R is a coalition such that $R \subset T_0$ and $\mu(R) > 0$. We adapt to our framework the argument used by Shitovitz (1973) to prove his Lemma 4. Let A denote the set of all rational numbers in the interval $(0, 1)$ and let $R_a = \{t \in R : u_t(a\check{\mathbf{x}}(t)) > u_t(\mathbf{w}(t))\}$. R_a is a coalition as the function $u(\cdot, \cdot)$ is measurable, by Assumption 3. Moreover, for each $t \in R$, there is an $a \in A$ such that $t \in R_a$ as $u_t(\cdot)$ is continuous, by Assumption 2. Then, it must be that $R = \bigcup_{a \in A} R_a$. But then, there exists an $a_0 \in A$ such that $\mu(R_{a_0}) > 0$ as μ is countably additive and $\mu(R) > 0$. Let $\bar{\mathbf{x}}$ be an assignment such that $\bar{\mathbf{x}}(m) = \check{\mathbf{x}}(m) + (1 - a_0) \int_{R_{a_0}} \check{\mathbf{x}}(t)\, d\mu$, $\bar{\mathbf{x}}(t) = a_0 \check{\mathbf{x}}(t)$, for each $t \in R_{a_0}$, and $\bar{\mathbf{x}}(t) = \check{\mathbf{x}}(t)$, for each $t \in T_0 \setminus R_{a_0}$. Then, it is straightforward to verify that $\bar{\mathbf{x}}$ satisfies the constraints of the Shitovitz maximization problem and $u_m(\bar{\mathbf{x}}(m)) > u_m(\check{\mathbf{x}}(m))$ as $\bar{\mathbf{x}}(m) \geq \check{\mathbf{x}}(m)$, $\bar{\mathbf{x}}(m) \neq \check{\mathbf{x}}(m)$, and $u_m(\cdot)$ is strongly monotone, by Assumption 2, a contradiction. Hence, it must be that $u_t(\check{\mathbf{x}}(t)) = u_t(\mathbf{w}(t))$, for each $t \in T_0$. ■

We now adapt to our framework the proof of the main theorem in Shitovitz (1997).

Proposition 16. *Under Assumptions 1, 2, and 3, if* $\check{\mathbf{x}}$ *is a solution to the Shitovitz maximization problem, then* $\check{\mathbf{x}}$ *is in the core.*

Proof. Let $\check{\mathbf{x}}$ be a solution to the Shitovitz maximization problem. Then, $\check{\mathbf{x}}$ is an allocation, by Corollary 4. We adapt to our framework the argument used by Shitovitz (1997) to prove his Proposition 2.3. Suppose that there is an allocation $\mathbf{y}$ which weakly dominates $\check{\mathbf{x}}$ via a coalition S such that $\{m\} \subset S$. Let $\bar{\mathbf{y}}$ be an assignment such that $\bar{\mathbf{y}}(t) = \mathbf{y}(t)$, for each $t \in S$ and $\bar{\mathbf{y}}(t) = \mathbf{w}(t)$, for each $t \in T \setminus S$. It is immediate to verify that $\bar{\mathbf{y}}$ satisfies the constraints of the Shitovitz maximization problem. Moreover, it must be that

$$u_m(\bar{\mathbf{y}})(m) = u_m(\mathbf{y})(m) > u_m(\check{\mathbf{x}})(m).$$

Then, $\check{\mathbf{x}}$ is not a solution to the Shitovitz maximization problem, a contradiction. Suppose that there is an allocation $\mathbf{y}$ which weakly dominates $\check{\mathbf{x}}$ via a coalition S such that $S \subseteq T_0$. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$. Then, we have that $u_t(\check{\mathbf{x}}(t)) = u_t((0, \mathbf{w}^2(t)))$, for each $t \in T_0$, by Corollary 5. Suppose that $\mathbf{y}^1(t) > 0$ for a non-null set of traders t in S. Then, it must be that $\int_S \mathbf{y}^1(t)\, d\mu > 0 = \int_S \mathbf{w}^1(t)$, a contradiction. But then, we have that $\mathbf{y}^2(t) > \mathbf{w}^2(t)$, as $u_t(\cdot)$ is strongly monotone, for each $t \in S$, by Assumption 2. This implies that $\int_S \mathbf{y}^2(t)\, d\mu > \int_S \mathbf{w}^2(t)\, d\mu$, a contradiction. Therefore, it must be that $\check{\mathbf{x}}$ is in the weak core. Hence, it is in the core, by Lemma 4 in Shitovitz (1973). ■

The following propositions express our version of the Shitovitz paradox.

Proposition 17. *Under Assumptions 1, 2, 3, and 4, let* $\mathbf{w}^i(m) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *be invertible on* R_{++} *and the function* $\mathbf{x}^0(t, \cdot)$ *be differentiable on* R_{++}, *for each* $t \in T_0$. *A solution* $\check{\mathbf{x}}$ *to the Shitovitz maximization problem is a monopoly allocation if and only if it is a Walras allocation.*

Proof. Assume, without loss of generality, that $\mathbf{w}^1(m) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t, \cdot)\, d\mu$ is invertible on R_{++}. Let $\check{\mathbf{x}}$ be a solution to the Shitovitz maximization problem. Then, $\check{\mathbf{x}}$ is an allocation, by Corollary 4. Suppose that $\check{\mathbf{x}}$ is a monopoly allocation. Then, $\check{\mathbf{x}}$ is in the core, by Proposition 16. Therefore, $\check{\mathbf{x}}$ is a Walras allocation, by Proposition 9. Suppose that $\check{\mathbf{x}}$ is a Walras allocation. Then, there is a price $\check{p}$ such that the pair $(\check{p}, \check{\mathbf{x}})$ is a Walras equilibrium. But then, there exists a strategy $\check{e} \in \mathbf{E}(m)$ such that $\check{e}_{12} = \mathbf{w}^1(m) - \check{\mathbf{x}}(m)$ and such that $\check{p} = \mathring{p}(\check{e})$, $\check{\mathbf{x}}(m) = \mathbf{x}(m, \check{e}, \mathring{p}(\check{e}))$, and $\check{\mathbf{x}}(t) = \mathbf{x}^0(t, \mathring{p}(\check{e}))$, for each $t \in T_0$, by the same argument used in the proof of Proposition 11. Let $e' \in \mathbf{E}(m)$ be a strategy such that $e' \neq \check{e}$ and let $\mathbf{x}'$

be an allocation such that $\mathbf{x}'(m) = \mathbf{x}(m, e', \mathring{p}(e'))$ and $\mathbf{x}'(t) = \mathbf{x}^0(t, \mathring{p}(e'))$, for each $t \in T_0$. It is straightforward to verify that the allocation $\mathbf{x}'$ satisfies the constraints of the Shitovitz maximization problem. Suppose that $u(\mathbf{x}'(m)) > u(\check{\mathbf{x}}(m))$. Then, $\check{\mathbf{x}}$ is not a solution to the Shitovitz maximization problem, a contradiction. Therefore, $\mathbf{x}'$ is a monopoly allocation as $\check{e} \in \mathbf{E}(m)$ is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Hence, a solution $\check{\mathbf{x}}$ to the Shitovitz maximization problem is a monopoly allocation if and only if it is a Walras allocation. ■

Proposition 18. *Under Assumptions 1, 2, 3, and 4, let $\mathbf{w}^i(m) > 0$ and let the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ be invertible on R_{++}, and the function $\mathbf{x}^0(t, \cdot)$ be differentiable on R_{++}, for each $t \in T_0$. If the set of monopoly allocations and the set of Walras allocations are disjoint and $\tilde{\mathbf{x}}$ is a monopoly allocation, then there is an allocation $\check{\mathbf{x}}$ in the core such that $u_m(\check{\mathbf{x}}(m)) > u_m(\tilde{x}(m))$ and $u_t(\check{\mathbf{x}}(t)) \leq u_t(\tilde{\mathbf{x}}(t))$, for each $t \in T_0$.*

Proof. Assume that $\mathbf{w}^i(m) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible on R_{++}, and the function $\mathbf{x}^0(t, \cdot)$ is differentiable on R_{++}, for each $t \in T_0$. Suppose that the set of monopoly allocations and the set of Walras allocations are disjoint. Let $\tilde{\mathbf{x}}$ be a monopoly allocation. Then, there exists a strategy $\tilde{e} \in \mathbf{E}(m)$ which is a monopoly equilibrium, with respect to the unique price selection $\mathring{p}(\cdot)$. Let $\tilde{p} = \mathring{p}(\tilde{e})$. $\tilde{\mathbf{x}}$ is not in the core as it is not a Walras allocation, by Proposition 9. Let $\check{\mathbf{x}}$ be a solution to the Shitovitz maximization problem. $\check{\mathbf{x}}$ is neither a monopoly allocation nor a Walras allocation, by Proposition 17. We have that $u_m(\check{\mathbf{x}}(m)) \geq u_m(\tilde{\mathbf{x}}(m))$, as $\tilde{\mathbf{x}}(m)$ satisfies the constraints of the Shitovitz maximization problem. Suppose that $u_m(\check{\mathbf{x}}(m)) = u_m(\tilde{\mathbf{x}}(m))$. Then, we have that $\tilde{\mathbf{x}}$ is in the core, by the same argument used in the proof of Proposition 16, a contradiction. But then, it must be that $u_m(\check{\mathbf{x}}(m)) > u_m(\tilde{\mathbf{x}}(m))$. Moreover, we have that $u_t(\check{\mathbf{x}}(t)) \leq u_t(\tilde{\mathbf{x}}(t))$ as $u_t(\tilde{\mathbf{x}}(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : \tilde{p}x = \tilde{p}\mathbf{w}(t)\}$, and $u_t(\check{\mathbf{x}}(t)) = u_t(\mathbf{w}(t))$, for each $t \in T_0$, by Corollary 5. ■

Example 2 in Section 5 exhibits a case where the set of monopoly allocations and the set of Walras allocations are disjoint and monopoly is advantageous. The next example shows that, for the same exchange economy, the Shitovitz paradox holds: there is an allocation in the core which is advantageous for the monopolist with respect to the monopoly allocation.

Example 2′. *Consider the exchange economy specified in Example 2. Then, the set of monopoly allocations and the set of Walras allocations are disjoint, there is a unique solution to the Shitovitz maximization problem $\check{\mathbf{x}}$ in* *the core such that $u_m(\check{\mathbf{x}}(m)) > u_m(\tilde{x}(m))$ and $u_t(\check{\mathbf{x}}(t)) \leq u_t(\tilde{\mathbf{x}}(t))$, for each $t \in T_0$, where $\tilde{\mathbf{x}}$ is the unique monopoly allocation.*

Proof. The set of monopoly allocations and the set of Walras allocations are disjoint as the allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(m) = (\frac{3}{4}, \frac{1}{4})$ and $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$, is the unique monopoly allocation and the allocation $\mathbf{x}^*$ such that $(\mathbf{x}^{*1}(m), \mathbf{x}^{*2}(m)) = (1 - (\frac{1}{4})^{\frac{1}{3}}, (\frac{1}{4})^{\frac{2}{3}})$, and $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = ((\frac{1}{4})^{\frac{1}{3}}, 1 - (\frac{1}{4})^{\frac{2}{3}})$, for each $t \in T_0$, is the unique Walras allocation, by Example 2. The core consists of all the allocations $\mathbf{x}$ of the form $(\mathbf{x}^1(m), \mathbf{x}^2(m)) = (\mathbf{x}^1(m), \frac{1}{4(1-\mathbf{x}^1(m))})$ and $(\mathbf{x}^1(t), \mathbf{x}^2(t)) = (1 - \mathbf{x}^1(m), 1 - \frac{1}{4(1-\mathbf{x}^1(m))})$, for each $t \in T_0$, where $1 - (\frac{1}{4})^{\frac{1}{3}} \leq \mathbf{x}^1(m) \leq 1 - (\frac{1}{4})^{\frac{2}{3}}$, as for such allocations there exists a price p such that the pair $(p, \mathbf{x})$ is an efficiency equilibrium and $p\mathbf{x}(t) \leq p\mathbf{w}(t)$, for each $t \in T_0$, by Theorem A^* in Shitovitz (1973). The allocation $\check{\mathbf{x}}$ such that $\check{\mathbf{x}}(m) = (1 - (\frac{1}{4})^{\frac{2}{3}}, (\frac{1}{4})^{\frac{1}{3}})$ and $\check{\mathbf{x}}(t) = ((\frac{1}{4})^{\frac{2}{3}}, 1 - (\frac{1}{4})^{\frac{1}{3}})$, for each $t \in T_0$, is the unique solution to the Shitovitz maximization problem and it is in the core. Moreover, we have that

$$u_m(\check{\mathbf{x}}(m)) = u_m(1-(\frac{1}{4})^{\frac{2}{3}}, (\frac{1}{4})^{\frac{1}{3}}) = \frac{1}{2}+\frac{3}{8}(\frac{1}{4})^{-\frac{1}{3}} > \frac{7}{8} = u_m\left(\frac{3}{4}, \frac{1}{4}\right) = u_m(\tilde{\mathbf{x}}(m)),$$

and

$$u_t(\check{\mathbf{x}}(t)) = u_t((\frac{1}{4})^{\frac{2}{3}}, 1 - (\frac{1}{4})^{\frac{1}{3}}) = 1 < \frac{5}{4} = u_t\left(\frac{1}{4}, \frac{3}{4}\right) = u_t(\tilde{\mathbf{x}}(t)),$$

for each $t \in T_0$. ■

7 Conclusion

In this paper, we have used the model of monopoly introduced by Busetto et al. (2023) to recast the relation between the economic welfare standard for antitrust and the explicit monopoly and perfectly competitive solutions in an Edgeworth box economy in which one commodity is held only by the monopolist – represented as an atom – and the other is held only by small traders – represented by an atomless part. In this framework, we have reconciled the approach characterizing the so-called Chicago School based on the notions of allocative efficiency and consumer welfare with the antimonopoly credo about "the curse of bigness" of the so-called New Brandeis School. Moreover, we have reformulated a paradox, due to Shitovitz (1997), which shows that the Brandeisian “curse of bigness” transcends the monopolists exercise of market power.

Busetto et al. (2023) also proposed a bilateral exchange version of the pioneering model of partial monopoly proposed by Forchheimer (1908), where a monopolist shares a market with a competitive fringe. We leave for further research an analysis of the antitrust implications of this model and a comparison with the results obtained in this paper.

Busetto et al. (2020) considered a bilateral oligopoly version of the Shapley window model with large traders, represented as atoms, and small traders, represented by an atomless part. It seems worthy of future research also an analysis of antitrust in this oligopolistic Edgeworth box economy and a comparison with the results obtained in this paper.

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