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Noncooperative Oligopoly in Markets with a Continuum of Traders and a Strongly Connected Set of Commodities: A Limit Theorem

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Francesca Busetto[*], Giulio Codognato[†], Sayantan Ghosal[‡], Ludovic Julien[§], Damiano Turchet[¶]

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Abstract

We consider a mixed version of the Shapley window model, where large traders are represented as atoms and small traders are represented by an atomless part. Motivated by the result that a countable infinity of atoms is neither a necessary nor a sufficient condition for a Cournot-Nash allocation to be a Walras allocation, we analyze the asymptotic relationship between the set of the Cournot-Nash allocations of the strategic market game and the Walras allocations of the exchange economy with which it is associated. Our main theorem shows that any sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of the exchange economy has a limit point for each trader and that the assignment determined by these limit points is a Walrasian allocation of the original economy. Instead of relying on restrictive assumptions on the characteristics of atoms, as in Busetto et al. (2017), our limit theorem relies

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on the characteristics of agents in the atomless part and their endogenously price-taking behavior.
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1 Introduction

Busetto et al. (2011) proved the existence of a Cournot-Nash equilibrium for the Shapley window model in mixed exchange economies à la Shitovitz, where large traders are represented as atoms and small traders are represented by an atomless part (see Shitovitz (1973)). The Shapley window model belongs to a very fruitful line of research on noncooperative market games, initiated by Lloyd S. Shapley and Martin Shubik (for a survey of this literature, see Giraud (2003)). The model was informally introduced by Lloyd S. Shapley and subsequently formalized by Sahi and Yao (1989) in the case of exchange economies with a finite number of traders. For this case, the authors proved the existence of a Cournot-Nash equilibrium. The proof provided by Busetto et al. (2011) for the mixed market case is based on the same assumptions used by Sahi and Yao (1989) for the finite case. In particular, it is required that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space. These restrictions are stated by Busetto et al. (2011) in their Assumption 4.

Busetto et al. (2017) analyzed the asymptotic behavior of the Cournot-Nash equilibria of the mixed version of the Shapley window model. They introduced a concept of replication which they called *à la* Cournot, since it extends to a general equilibrium context the original Cournotian idea of replication: it consists in partially replicating the economy by increasing only the number of atoms, this way making them asymptotically negligible, without affecting the atomless part. Under the same assumptions of the existence theorem proved by Busetto et al. (2011), these authors proved a theorem establishing that any sequence of Cournot-Nash allocations of the strategic market games associated with the partial replications of the exchange economy has a limit point for each trader and that the assignment determined by these limit points is a Walrasian allocation of the original economy.

Busetto et al. (2018) proved a new existence theorem for the mixed version of the Shapley window model, differing from the one proposed by

Busetto et al. (2011) in that it is essentially based on restrictions on endowments and preferences of the atomless part of the economy rather than of atoms. In particular, Busetto et al. (2018) removed Assumption 4 in Busetto et al. (2011) and used the fact – proved by Codognato and Ghosal (2000) – that traders belonging to the atomless part have an endogenous "Walrasian" behavior. In the work of 2018, this property was exploited to show that, under the assumptions that each commodity is held, in the aggregate, by the atomless part and that traders' utility functions are continuous, strongly monotone, quasi-concave, and measurable, any sequence of prices corresponding to a sequence of Cournot-Nash equilibria has a subsequence which converges to a strictly positive price vector. The authors used this price convergence result to prove their existence theorem, under the assumption that the set of commodities is strongly connected through traders' characteristics, which imposes a joint restriction on the endowments and preferences of the atomless part and is a variant of a hypothesis first proposed by Codognato and Ghosal (2000). This assumption, combined with the continuity properties of the Walrasian correspondence generated by the atomless part's behavior, in turn guarantees that the aggregate matrix of the bids obtained as the limit of a sequence of perturbed Cournot-Nash equilibria is irreducible.

In this paper, we consider the mixed version of the Shapley window model in the formulation proposed by Busetto et al. (2018), with the aim of establishing the asymptotic properties of its equilibria. We use the same concept of replication *à la* Cournot introduced by Busetto et al. (2017) to show a new limit theorem which does not require the restrictions on atoms stated in their Assumption 4. The proof of the new limit result rests heavily on the price convergence theorem shown by Busetto et al. (2018). As a consequence, that result turns out to be merely explained, like those authors' existence theorem, in terms of the characteristics of the atomless part of the economy and the fact that the traders belonging to it have an exogenous "Walrasian" behavior.

Following Busetto et al. (2017) and Codognato et al. (2015), we provide two examples which show that the condition that an economy contains a countably infinite number of atoms is neither necessary nor sufficient to guarantee that any Cournot-Nash allocation is a Walras allocation.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we restate the price convergence theorem. In Section 4, we introduce the replication *à la Cournot*. In Section 5, we prove the existence of an atom-type-symmetric Cournot-Nash equilibrium.

In Section 6, we state and prove the limit theorem. In Section 7, we discuss the model. In Section 8, we draw some conclusions from our analysis.

2 Mathematical model

We consider an exchange economy, $\mathcal{E}$, with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < +\infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and $T_0 = T \setminus T_1$ the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are l different commodities. A commodity bundle is a point in R^l_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^l_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}(t) > 0$, *for each* $t \in T$, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^l_+ \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R^l_+ \to R$ *is continuous, strongly monotone, and quasi-concave, for each* $t \in T$.

Let $\mathcal{B}$ denote the Borel σ-algebra of R^l_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by all the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times R^l_+ \to R$, *given by* $u(t, x) = u_t(x)$, *for each* $t \in T$ *and for each* $x \in R^l_+$, *is* $\mathcal{T} \bigotimes \mathcal{B}$*-measurable.*

In order to state our last assumption, we need some preliminary definitions. We denote by L the set of commodities $\{1, \ldots, l\}$. We say that two commodities $i, j \in L$ stand in relation C if there is a measurable set T^i, with $\mu(T^i) > 0$, such that $T^i = \{t \in T_0 : \mathbf{w}^i(t) > 0,\ \mathbf{w}^r(t) = 0,$ for each $r \in L \setminus \{i\}\}$, $u_t(\cdot)$ is differentiable, additively separable in commodity j, i.e.,

$u_t(x) = v_t^j(x^j) + v_t(x^1, \dots, x^{j-1}, x^{j+1}, \dots, x^l)$, for each $x \in R_+^l$, and $\frac{dv_t^j(0)}{dx^j} = +\infty$, for each $t \in T^i$.[1] Then, the concept of a set of commodities strongly connected through traders' characteristics can be defined as follows.

Definition 1. *The set of commodities L is said to be strongly connected through traders' characteristics if $\{(i,j) : iCj\} \neq \emptyset$ and the directed graph $D_L(L, C)$ is strongly connected, i.e., any ordered pair of distinct vertices, i and j, of $D_L(L, C)$ is connected by a path.*

We can now state our last assumption.

Assumption 4. *The set of commodities L is strongly connected through traders' characteristics.*

A price vector is a nonnull vector $p \in R_+^l$. Henceforth, we say that a price vector p is normalized if $p \in \Delta$, where $\Delta = \{p \in R_+^l : \sum_{i=1}^l p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

Let $\mathbf{X}^0 : T_0 \times \Delta \setminus \partial\Delta \to \mathcal{P}(R^l)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R_{++}^l$, $\mathbf{X}^0(t, p) = \text{argmax}\{u(x) : x \in R_+^l \text{ and } px \leq p\mathbf{w}(t)\}$. It is well-known that the previous assumptions guarantee that the correspondence $\mathbf{X}^0(t, \cdot)$ is upper hemicontinuous, for each $t \in T_0$.

A Walras equilibrium of $\mathcal{E}$ is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector $p^* \in \Delta \setminus \partial\Delta$ and an allocation $\mathbf{x}^*$, such that, for each $t \in T$, $u_t(\mathbf{x}^*(t)) \geq u_t(y)$, for all $y \in \{x \in R_+^l : p^*x = p^*\mathbf{w}(t)\}$. A Walras allocation of $\mathcal{E}$ is an allocation $\mathbf{x}^*$ for which there exists a price vector $p^* \in \Delta \setminus \partial\Delta$ such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium of $\mathcal{E}$.

We define now the strategic market game, Γ, associated with $\mathcal{E}$. It is a slightly reformulated version of the Shapley window model for mixed economies proposed by Busetto et al. (2011).

A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R_+^{l^2})$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R_+^{l^2} : \sum_{j=1}^l b_{ij} \leq \mathbf{w}^i(t),\ i = 1, \dots, l\}$. With some abuse of notation, we denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j = 1, \dots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to R_+^{l^2}$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}$ to be the

[1] In this definition, differentiability is to be understood as continuous differentiability and it includes the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58). Moreover, it can be proved that the separable utility function used in the definition is the representation of separable preferences (see, for instance, Kreps (2012), p. 42).

matrix such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j = 1, \dots, l$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$, and by $\bar{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

The following definitions are borrowed from Sahi and Yao (1989).

Definition 2. *A nonnegative square matrix A is said to be irreducible if, for every pair (i, j), with $i \neq j$, there is a positive integer k such that $a_{ij}^{(k)} > 0$, where $a_{ij}^{(k)}$ denotes the ij-th entry of the k-th power A^k of A.*

Definition 3. *Given a strategy selection $\mathbf{b}$, a normalized price vector p is said to be market clearing if*

$$p \in \Delta \setminus \partial\Delta, \ \sum_{i=1}^{l} p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^{l} \bar{\mathbf{b}}_{ji}), \ j = 1, \dots, l. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique normalized price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique normalized price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0, otherwise.

Given a strategy selection $\mathbf{b}$ and a normalized price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{b}_{ji}(t) + \sum_{i=1}^{l} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in \Delta \setminus \partial\Delta, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,}
\end{aligned}$$

$j = 1, \dots, l$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, the traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$.[2] It is straightforward to show that this assignment is an allocation.

We are now able to introduce a notion of Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal (2000) and Busetto et al. (2011)).

[2] In order to save in notation, with some abuse we denote by $\mathbf{x}$ both the function $\mathbf{x}(t)$ and the function $\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b}))$.

Definition 4. *A strategy selection* $\hat{\mathbf{b}}$ *such that* $\bar{\hat{\mathbf{B}}}$ *is irreducible is a Cournot-Nash equilibrium of* Γ *if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ *and for each* $t \in T$.[3]

A Cournot-Nash allocation of Γ is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium of Γ.

3 Price convergence theorem

By means of Lemma 9 in Sahi and Yao (1989), Busetto et al. (2011) showed that any convergent sequence of normalized prices corresponding to a sequence of Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive normalized price vector. Lemma 9 in Sahi and Yao (1989), and consequently the convergence result in Busetto et al. (2011), are essentially based on the assumption that there are at least two atoms with strictly positive endowments, continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space. This restriction is stated by Busetto et al. (2011) in their Assumption 4.

Busetto et al. (2017) also used Lemma 9 in Sahi and Yao (1989) to prove their limit theorem under the same assumption.

Busetto et al. (2018) provided a different price convergence theorem, obtained by removing Assumption 4 in Busetto et al. (2011) and focusing on restrictions concerning endowments and preferences of the atomless part of the economy rather than of atoms. More precisely, they exploited the property of small traders, proved by Codognato and Ghosal (2000), of being "Walrasian" at a Cournot-Nash equilibrium. Their price convergence theorem establishes that any sequence of normalized prices corresponding to a sequence of Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive normalized price vector. They used it to show their main existence theorem. Here, we use it to prove our new limit theorem for mixed exchange economies where Assumption 4 in Busetto et al. (2011) is relaxed.

[3]Let us notice that, as this definition of a Cournot-Nash equilibrium explicitly refers to irreducible matrices, it applies only to active equilibria (on this point, see Sahi and Yao (1989)).

For the seek of convenience, we repropose the formal statement of the price convergence theorem shown by Busetto et al. (2018).

Theorem 1. *Under Assumptions 1, 2, and 3, let $\{\hat{p}^n\}$ be a sequence of normalized prices such that $\{\hat{p}^n\} = p(\hat{\mathbf{b}}^n)$ where $\hat{\mathbf{b}}^n$ is a Cournot-Nash equilibrium of Γ, for each $n = 1, 2, \ldots$. Then, there exists a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to a price vector $\hat{p} \in \Delta \setminus \partial\Delta$.*

Proof. See the proof of Theorem 1 in Busetto et al. (2018). ■

4 Replication à la Cournot of $\mathcal{E}$

In this section, we focus on the concept of replication introduced by Busetto et al. (2017), in the original spirit of Cournot (1838). We will use this concept to obtain our limit theorem for the Cournot-Nash equilibria of the mixed version of the Shapley window model. By analogy with the replication proposed by Cournot in a partial equilibrium framework, the concept proposed by Busetto et al. (2017) is obtained by replicating only the atoms of $\mathcal{E}$, while making them asymptotically negligible, and without affecting the atomless part.

This replication *à la Cournot* of $\mathcal{E}$ can be formalized as follows. Let $\mathcal{E}^n$ be an exchange economy characterized as in Section 2, where each atom is replicated n times. For each $t \in T_1$, let tr denote the r-th element of the n-fold replication of t. We assume that, for each $t \in T_1$, $\mathbf{w}(tr) = \mathbf{w}(ts) = \mathbf{w}(t)$, $u_{tr}(\cdot) = u_{ts}(\cdot) = u_t(\cdot)$, $r, s = 1, \ldots, n$, and $\mu(tr) = \frac{\mu(t)}{n}$, $r = 1, \ldots, n$. Clearly, $\mathcal{E}^1$coincides with $\mathcal{E}$.

Then, the strategic market game Γ^n associated with $\mathcal{E}^n$ can be characterized, *mutatis mutandis*, as in Section 2. Clearly, Γ^1 coincides with Γ. A strategy selection $\mathbf{b}$ of Γ^n is said to be atom-type-symmetric if $\mathbf{b}^n(tr) = \mathbf{b}^n(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$.

We provide now the definition of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n.

Definition 5. *A strategy selection $\hat{\mathbf{b}}$ such that $\bar{\hat{\mathbf{B}}}$ is irreducible is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n if $\hat{\mathbf{b}}$ is atom-type-symmetric and if*

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}(tr), p(\hat{\mathbf{b}}))) \geq u_{tr}(\mathbf{x}(tr, b(tr), p(\hat{\mathbf{b}} \setminus b(tr)))),$$

for each $b(tr) \in \mathbf{B}(tr)$, $r = 1, \ldots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T_0$.

In order to show the existence of an atom type-symmetric Cournot-Nash equilibrium of Γ^n, we need to define the notion of a perturbation of this strategic market game, denoted by $\Gamma^n(\epsilon)$ (it was already used by Sahi and Yao (1989), Busetto et al. (2011), and Busetto et al. (2018)).

Given $\epsilon > 0$ and a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}}_\epsilon$ to be the matrix such that $\bar{\mathbf{b}}_{\epsilon ij} = (\bar{\mathbf{b}}_{ij} + \epsilon)$, $i, j = 1, \ldots, l$. Clearly, the matrix $\bar{\mathbf{B}}_\epsilon$ is irreducible. The interpretation is that an outside agency places fixed bids of ϵ for each pair of commodities (i, j).

Given $\epsilon > 0$, we denote by $p^\epsilon(\mathbf{b})$ the function which associates, with each strategy selection $\mathbf{b}$, the unique normalized price vector which satisfies

$$\sum_{i=1}^{l} p^i(\bar{\mathbf{b}}_{ij} + \epsilon) = p^j(\sum_{i=1}^{l} \bar{\mathbf{b}}_{ji} + \epsilon)),\ j = 1, \ldots, l.$$

Then, let us introduce the following notion of equilibrium for $\Gamma^n(\epsilon)$.

Definition 6. *Given $\epsilon > 0$, a strategy selection $\hat{\mathbf{b}}^\epsilon$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$ if $\hat{\mathbf{b}}^\epsilon$ is atom-type-symmetric and*

$$u_{tr}(\mathbf{x}(tr, \hat{\mathbf{b}}^\epsilon(tr), p^\epsilon(\hat{\mathbf{b}}^\epsilon))) \geq u_{tr}(tr, b(tr), p^\epsilon(\hat{\mathbf{b}}^\epsilon \setminus b(tr)))),$$

for each $b(tr) \in \mathbf{B}(tr)$, $r = 1, \ldots, n$, and for each $t \in T_1$;

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}^\epsilon(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon))) \geq u_t(t, b(t), p^\epsilon(\hat{\mathbf{b}}^\epsilon \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T_0$.

5 Existence of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n

The theorem presented in this section establishes the existence of an atom-type-symmetric Cournot-Nash equilibrium of Γ^n. The proof of the theorem differs from that provided by Busetto et al. (2017) in that it replaces their Assumption 4 on endowments and preferences of atoms (the same as Assumption 4 in Busetto et al. (2011)) with the assumption that the set of commodities is strongly connected through traders' characteristics, which imposes restrictions on endowments and preferences of the atomless part. Our existence theorem is based on that proved by Busetto et al. (2018), which rests crucially on the price converges theorem presented in Section 3.

Theorem 2. *Under Assumptions 1, 2, 3, and 4, there exists an atom-type-symmetric Cournot-Nash equilibrium* $\hat{\mathbf{b}}$ *of* Γ^n.

Proof. We first need to prove the existence of an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$. To do so, we apply, as in Busetto et al. (2011), the Kakutani-Fan-Glicksberg theorem.

We neglect, as usual, the distinction between integrable functions and equivalence classes of such functions and denote by $L_1(\mu, R^{l^2})$ the set of integrable functions taking values in R^{l^2}, by $L_1(\mu, \mathbf{B}(\cdot))$ the set of strategy selections, and by $L_1(\mu, \mathbf{B}^*(\cdot))$ the set of atom-type-symmetric strategy selections. Note that the locally convex Hausdorff space we shall be working in is $L_1(\mu, R^{l^2})$, endowed with its weak topology.

The proof of existence of an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$ is articulated in three lemmas.

The first lemma establishes the properties of $L_1(\mu, \mathbf{B}^*(\cdot))$ required to apply the Kakutani-Fan-Glicksberg theorem.

Lemma 1. *Under Assumptions 1, 2, 3, and 4, the set* $L_1(\mu, \mathbf{B}^*(\cdot))$ *is nonempty, convex and weakly compact.*

Proof. See the proof of Lemma 1 in Busetto et al. (2017). ■

Given $\epsilon > 0$, let $\alpha^\epsilon_{tr} : L_1(\mu, \mathbf{B}^*(\cdot)) \to \mathbf{B}(tr)$ be a correspondence such that $\alpha^\epsilon_{tr}(\mathbf{b}) = \operatorname{argmax}\{u_{tr}(\mathbf{x}(t, b(tr), p^\epsilon(\mathbf{b} \setminus b(tr)))) : b(tr) \in \mathbf{B}(tr)\}$, $r = 1, \ldots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, and let $\alpha^\epsilon_t : L_1(\mu, \mathbf{B}(\cdot)) \to \mathbf{B}(t)$ be a correspondence such that $\alpha^\epsilon_t(\mathbf{b}) = \operatorname{argmax}\{u_t(\mathbf{x}(t, b(t), p^\epsilon(\mathbf{b}\setminus b(t)))) : b(t) \in \mathbf{B}(t)\}$, for each $t \in T_0$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Moreover, let $\alpha^\epsilon : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}(\cdot))$ be a correspondence such that $\alpha^\epsilon(\mathbf{b}) = \{\mathbf{b} \in L_1(\mu, \mathbf{B}(\cdot)) : \mathbf{b}(tr) \in \alpha^\epsilon_{tr}(\mathbf{b}),\ r = 1, \ldots, n,$ for each $t \in T_1$, and $\mathbf{b}(t) \in \alpha^\epsilon_t(\mathbf{b})$, for each $t \in T_0\}$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Finally, let $\alpha^{\epsilon*} : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}^*(\cdot))$ be a correspondence such that $\alpha^{\epsilon*}(\mathbf{b}) = \alpha^\epsilon(\mathbf{b}) \cap L_1(\mu, \mathbf{B}^*(\cdot))$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$.

The second lemma provides us with the properties of the correspondence $\alpha^{\epsilon*}$.

Lemma 2. *Under Assumptions 1, 2, 3, and 4, given* $\epsilon > 0$, *the correspondence* $\alpha^{\epsilon*}$ *is nonempty, convex-valued, and it has a weakly closed graph.*

Proof. Let $\epsilon > 0$ be given. We have that $\alpha^\epsilon_{tr}(\mathbf{b})$ is nonempty, $r = 1, \ldots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, by the argument used in the proof of Lemma 2 in Busetto et al. (2011). Moreover, we have that $\alpha^\epsilon_{tr}(\mathbf{b}) = \alpha^\epsilon_{ts}(\mathbf{b})$ as $u_{tr}(\cdot) = u_{ts}(\cdot)$ and $\mathbf{B}(tr) = \mathbf{B}(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. Then, there exists a strategy $\tilde{b}(t) \in \mathbf{B}(t)$ such that $\tilde{b}(t) \in \alpha^{\epsilon}_{tr}(\mathbf{b})$, $r = 1, \ldots, n$, for each $t \in T_1$ and for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. But then, $\alpha^{\epsilon *}(\mathbf{b})$ is nonempty, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$, by the same argument used in the proof of Lemma 2 in Busetto et al. (2011). $\alpha^{\epsilon *}(\mathbf{b})$ is convex as $\alpha^{\epsilon}(\mathbf{b})$ is convex, by Lemma 2 in Busetto et al. (2011), and $L_1(\mu, \mathbf{B}^*(\cdot))$ is convex, by Lemma 1, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. α^{ϵ} has a weakly closed graph, by Lemma 2 in Busetto et al. (2011). Let $\phi : L_1(\mu, \mathbf{B}^*(\cdot)) \to L_1(\mu, \mathbf{B}^*(\cdot))$ be a correspondence such that $\phi(\mathbf{b}) = L_1(\mu, \mathbf{B}^*(\cdot))$, for each $\mathbf{b} \in L_1(\mu, \mathbf{B}^*(\cdot))$. It is straightforward to verify that ϕ has a weakly closed graph. Then, $\alpha^{\epsilon *}$ has a weakly closed graph as it is the intersection of the weakly closed correspondences α^{ϵ} and ϕ, by Theorem 17.25 in Aliprantis and Border (2006). ∎

Finally, the third lemma proves the existence of an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$.

Lemma 3. *Under Assumptions 1, 2, 3, and 4, given $\epsilon > 0$, there exists an atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon}$ of $\Gamma^n(\epsilon)$.*

Proof. Let $\epsilon > 0$ be given. The set $L_1(\mu, \mathbf{B}^*(\cdot))$ is nonempty, convex and weakly compact, by Lemma 1. Moreover, the correspondence $\alpha^{\epsilon *}$ is nonempty, convex-valued, and it has a weakly closed graph, by Lemma 2. Then, there exists a fixed point $\hat{\mathbf{b}}^{\epsilon}$ of the correspondence $\alpha^{\epsilon *}$ by the Kakutani-Fan-Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Border (2006)). Hence, $\hat{\mathbf{b}}^{\epsilon}$ is an atom-type-symmetric ϵ-Cournot-Nash equilibrium of $\Gamma^n(\epsilon)$. ∎

To complete the proof of Theorem 2, we have to show that there exists the limit of a sequence of atom-type-symmetric ϵ-Cournot-Nash equilibria of $\Gamma^n(\epsilon)$ and that this limit is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n. Following Busetto et al. (2011), in this part of the proof we essentially refer to a generalization of the Fatou's lemma in several dimensions provided by Artstein (1979). Let $\epsilon_m = \frac{1}{m}$, $m = 1, 2, \ldots$. By Lemma 3, for each $m = 1, 2, \ldots$, there is an atom-type-symmetric ϵ-Cournot-Nash equilibrium $\hat{\mathbf{b}}^{\epsilon_m}$ of $\Gamma^n(\epsilon)$. The facts that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_m}\}$ belongs to the compact set $\{(b_{ij}) \in R^{l^2}_{+} : b_{ij} \leq n \int_{T_1} \mathbf{w}^i(t)\, d\mu + \int_{T_0} \mathbf{w}^i(t)\, d\mu \; i, j = 1, \ldots, l\}$, the sequence $\{\hat{\mathbf{b}}^{\epsilon_m}(tr)\}$ belongs to the compact set $\mathbf{B}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, and the sequence $\{\hat{p}^{\epsilon_m}\}$, where $\hat{p}^{\epsilon_m} = p^{\epsilon_m}(\hat{\mathbf{b}}^{\epsilon_m})$, for each $m = 1, 2, \ldots$, belongs to the unit simplex Δ, imply that there is a subsequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_m}}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_m}\}$ which converges to an element of the set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq n \int_{T_1} \mathbf{w}^i(t)\, d\mu + \int_{T_0} \mathbf{w}^i(t)\, d\mu,\ i, j = 1, \ldots, l\}$, a subsequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$ of the sequence $\{\hat{\mathbf{b}}^{\epsilon_m}(tr)\}$ which converges to an element of the set $\mathbf{B}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, and a subsequence $\{\hat{p}^{\epsilon_{k_m}}\}$ of the sequence $\{\hat{p}^{\epsilon_m}\}$ which converges to a price vector $\hat{p} \in \Delta \setminus \partial\Delta$, by Theorem 1. Since the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(tr)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(t)\}$, for each $t \in T_0$, and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{\epsilon_{k_m}}\}$ converges to $\bar{\hat{\mathbf{B}}}$. Then, $\hat{\mathbf{b}}(tr) = \hat{\mathbf{b}}(ts)$ as $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\} = \{\hat{\mathbf{b}}^{\epsilon_{k_m}}(ts)\}$, $r, s = 1, \ldots, n$, for each $t \in T_1$, and $\hat{\mathbf{b}}(tr)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{\epsilon_{k_m}}(tr)\}$, $r = 1, \ldots, n$, for each $t \in T_1$. Hence, it can be proved, by the same argument used by Busetto et al. (2018) to show their existence theorem, that $\hat{\mathbf{b}}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n.

■

6 Limit theorem

In this section, we state and prove our limit theorem. It establishes that, given a sequence of atom-type-symmetric Cournot-Nash allocations of Γ^n, for $n = 1, 2, \ldots$, there exists a Walras allocation of $\mathcal{E}$ with the following property: for each trader $t \in T$, the value of this Walras allocation at t is a limit point of the sequence of final holdings of t associated with the sequence of atom-type-symmetric Cournot-Nash equilibria of Γ^n, for $n = 1, 2, \ldots$.

Theorem 3. *Under Assumptions 1, 2, 3, and 4, let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1, 2, \ldots$. Then,*
(i) there exists a subsequence $\{\hat{\mathbf{b}}^{k_n}\}$ of the sequence $\{\hat{\mathbf{b}}^n\}$, a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$, a strategy selection $\hat{\mathbf{b}}$ of Γ, and a price vector $\hat{p} \in \Delta \setminus \partial\Delta$, such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, the sequence $\{\bar{\hat{\mathbf{B}}}^{k_n}\}$ converges to $\bar{\hat{\mathbf{B}}}$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$;
(ii) $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_1$, and $\hat{\mathbf{x}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_0$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ for each $t \in T$, and $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, for each $t \in T$, and for $n = 1, 2, \ldots$;

(iii) the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of $\mathcal{E}$.

Proof. (i) Let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1, 2, \ldots$. The facts that the sequence $\{\bar{\hat{\mathbf{B}}}^n\}$ belongs to the compact set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\, i, j = 1, \ldots, l\}$, the sequence $\{\hat{\mathbf{b}}^n(t)\}$ belongs to the compact set $\mathbf{B}(t)$, for each $t \in T_1$, and the sequence $\{\hat{p}^n\}$, belongs to the unit simplex Δ, imply that there is a subsequence $\{\bar{\hat{\mathbf{B}}}^{k_n}\}$ of the sequence $\{\bar{\hat{\mathbf{B}}}^n\}$ which converges to an element of the set $\{(b_{ij}) \in R_+^{l^2} : b_{ij} \leq \int_T \mathbf{w}^i(t)\, d\mu,\, i, j = 1, \ldots, l\}$, a subsequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$ of the sequence $\{\hat{\mathbf{b}}^n(t)\}$ which converges to an element of the set $\mathbf{B}(t)$, for each $t \in T_1$, and a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$ which converges to a price vector $\hat{p} \in \Delta \setminus \partial\Delta$, by Theorem 1. Since the sequence $\{\hat{\mathbf{b}}^{k_n}\}$ satisfies the assumptions of Theorem A in Artstein (1979), there is a function $\hat{\mathbf{b}}$ such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, and such that the sequence $\{\bar{\hat{\mathbf{B}}}^{k_n}\}$ converges to $\bar{\hat{\mathbf{B}}}$.
(ii) Let $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ for each $t \in T$, and $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$, for each $t \in T$, and for $n = 1, 2, \ldots$. Then, $\hat{\mathbf{x}}(t)$ is the limit of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_1$, as $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$, $\hat{\mathbf{x}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{x}}^{k_n}(t)\}$, for each $t \in T_0$, as $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$.
(iii) $\bar{\hat{\mathbf{B}}}^{\Gamma^n} = \bar{\hat{\mathbf{B}}}^n$ as $\bar{\hat{\mathbf{b}}}_{ij}^{\Gamma^n} = \sum_{t \in T_1} \sum_{r=1}^n \hat{\mathbf{b}}_{ij}^{\Gamma^n}(tr)\mu(tr) + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^{\Gamma^n}(t)\, d\mu = \sum_{t \in T_1} n\hat{\mathbf{b}}_{ij}^n(t)\frac{\mu(t)}{n} + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^n(t)\, d\mu = \sum_{t \in T_1} \hat{\mathbf{b}}_{ij}^n(t)\mu(t) + \int_{t \in T_0} \hat{\mathbf{b}}_{ij}^n(t)\, d\mu = \bar{\hat{\mathbf{b}}}_{ij}^n$, $i, j = 1, \ldots, l$, for $n = 1, 2, \ldots$. Then, $\hat{p}^n = p(\hat{\mathbf{b}}^n)$ as $\hat{p}^n$ and $\hat{\mathbf{b}}^n$ satisfy (1), for $n = 1, 2, \ldots$. But then, by continuity, $\hat{p}$ and $\hat{\mathbf{b}}$ must satisfy (1). We now show that, if two commodities $i, j \in L$ stand in the relation C, then $\bar{\hat{\mathbf{b}}}_{ij} > 0$. Suppose that $\bar{\hat{\mathbf{b}}}_{ij} = 0$. Then, $\int_{T^i} \hat{\mathbf{b}}_{ij}(t)\, d\mu = 0$ as $\mu(T^i) > 0$. Consider a trader $\tau \in T^i$. We can suppose that $\hat{\mathbf{b}}_{ij}(\tau) = 0$ as we ignore null sets. Since $\hat{\mathbf{b}}(\tau)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$, there is a subsequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ of this sequence which converges to $\hat{\mathbf{b}}(\tau)$. Then, the subsequence $\{\hat{\mathbf{x}}^{h_{k_n}}(\tau)\}$ of the sequence $\{\hat{\mathbf{x}}^{k_n}(\tau)\}$ converges to $\hat{\mathbf{x}}(\tau)$ as the sequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$ and the sequence $\{\hat{p}^{h_{k_n}}\}$ converges to $\hat{p}$. But then, we have that $\hat{\mathbf{x}}^j(\tau) = 0$ as $\hat{\mathbf{b}}_{ij}(\tau) = 0$ and $\hat{\mathbf{x}}(\tau) \in \mathbf{X}^0(\tau, \hat{p})$

as $\hat{\mathbf{x}}^{h_{k_n}}(\tau) \in \mathbf{X}^0(\tau, \hat{p}^{h_{k_n}})$, for each $n = 1, 2, \ldots$, by the same argument used by Codognato and Ghosal (2000) to prove their Theorem 2, and the correspondence $\mathbf{X}^0(\tau, \cdot)$ is upper hemicontinuous. Therefore, we have that $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^j} = +\infty$ as $i, j \in L$ stand in the relation C and $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^j} \leq \nu \hat{p}^j$, by the necessary conditions of the Kuhn-Tucker Theorem. Moreover, there must be a commodity h such that $\hat{\mathbf{x}}^h(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $\hat{p}\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^h} = \nu \hat{p}^h$, by the necessary conditions of the Kuhn-Tucker Theorem. But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^h} = +\infty$ as $\nu = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, if two commodities $i, j \in L$ stand in the relation C, then $\bar{\hat{\mathbf{b}}}_{ij} > 0$. This implies that the matrix $\bar{\hat{\mathbf{B}}}$ is irreducible by our Assumption 4 and by the argument used by Codognato and Ghosal (2000) in the proof of their Theorem 2. Consider the pair $(\hat{p}, \hat{\mathbf{x}})$. It is straightforward to show that the assignment $\hat{\mathbf{x}}$ is an allocation as $\hat{p}$ and $\hat{\mathbf{b}}$ satisfy (1) and that $\hat{\mathbf{x}}(t) \in \{x \in R^l_+ : \hat{p}x = \hat{p}\mathbf{w}(t)\}$, for each $t \in T$. Suppose that $(\hat{p}, \hat{\mathbf{x}})$ is not a Walras equilibrium of $\mathcal{E}$. Then, there exists a trader $\tau \in T$ and a commodity bundle $\tilde{x} \in \{x \in R^l_+ : \hat{p}x = \hat{p}\mathbf{w}(\tau)\}$ such that $u_\tau(\tilde{x}) > u_\tau(\hat{\mathbf{x}}(\tau))$. By Lemma 5 in Codognato and Ghosal (2000), there exist real numbers $\tilde{\lambda}^j \geq 0$, with $\sum_{j=1}^l \tilde{\lambda}^j = 1$, such that

$$\tilde{x}^j = \tilde{\lambda}^j \frac{\sum_{i=1}^l \hat{p}^i \mathbf{w}^i(\tau)}{\hat{p}^j}, \; j = 1, \ldots, l.$$

Let $\tilde{b}_{ij} = \mathbf{w}^i(\tau)\tilde{\lambda}^j$, $i, j = 1, \ldots, l$. Then, it is straightforward to verify that

$$\tilde{x}^j = \mathbf{w}^j(\tau) - \sum_{i=1}^l \tilde{b}_{ji} + \sum_{i=1}^l \tilde{b}_{ij} \frac{\hat{p}^i}{\hat{p}^j},$$

for each $j = 1, \ldots, l$. Consider the following cases.

Case 1. $\tau \in T_1$. Let ρ denote the k_1-th element of the k_n-fold replication of $\mathcal{E}$ and let $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ be the aggregate matrix corresponding to the strategy selection $\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$, where $\tilde{b}(\tau\rho) = \tilde{b}$, for $n = 1, 2, \ldots$. Let $\Delta\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ and $\Delta\bar{\hat{\mathbf{B}}}^{k_n}$ denote the diagonal matrices of row sums of, respectively, $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ and $\bar{\hat{\mathbf{B}}}^{k_n}$, for $n = 1, 2, \ldots$. Moreover, let $q^{\Gamma^{k_n}}_{\tau\rho}$, and q^{k_n} denote the vectors of the cofactors of the first column of, respectively, $\Delta\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho) - \bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$ and $\Delta\bar{\hat{\mathbf{B}}}^{k_n} - \bar{\hat{\mathbf{B}}}^{k_n}$, for $n = 1, 2, \ldots$. Clearly, $q^{\Gamma^{k_n}} = q^{k_n}$ as $\bar{\hat{\mathbf{B}}}^{\Gamma^{k_n}} = \bar{\hat{\mathbf{B}}}^{k_n}$, for $n = 1, 2, \ldots$. Let $\Delta\bar{\hat{\mathbf{B}}}$ be the diagonal matrix of row sums of $\hat{\bar{\mathbf{B}}}$ and q be the cofactors of the first column of $\Delta\hat{\bar{\mathbf{B}}} - \hat{\bar{\mathbf{B}}}$. The sequence $\{q^{k_n}\}$ converges to q as the sequence $\hat{\bar{\mathbf{B}}}^{k_n}$ converges to $\hat{\bar{\mathbf{B}}}$. Let $\bar{w} = \max\{\mathbf{w}^1(\tau), \ldots, \mathbf{w}^l(\tau)\}$. Consider the matrix $\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)$, for $n = 1, 2, \ldots$. Then, we have that $\hat{\bar{\mathbf{b}}}_{ij}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{b}}}_{ij}^{\Gamma^{k_n}} \setminus \tilde{b}_{ij}(\tau\rho) = (\frac{1}{k_n}\hat{\mathbf{b}}_{ij}^{\Gamma^{k_n}}(\tau\rho) - \frac{1}{k_n}\tilde{b}_{ij}(\tau\rho))$, $i, j = 1, \ldots, l$, for $n = 1, 2, \ldots$. But then, the sequence of Euclidean distances $\{\|\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\|\}$ converges to 0 as $|\frac{1}{k_n}\hat{\mathbf{b}}_{ij}^{\Gamma^{k_n}}(\tau\rho) - \frac{1}{k_n}\tilde{b}_{ij}(\tau\rho)| = \frac{1}{k_n}|\hat{\mathbf{b}}_{ij}^{\Gamma^{k_n}}(\tau\rho) - \tilde{b}_{ij}(\tau\rho)| \leq \frac{1}{k_n}\bar{w}$, $i, j = 1, \ldots, l$, $n = 1, 2, \ldots$. The sequence $\{\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\}$ converges to $\hat{\bar{\mathbf{B}}}$ as, by the triangle inequality, $\|\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\} - \hat{\bar{\mathbf{B}}}\| \leq \|\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\| + \|\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{B}}}\| = \|\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\| + \|\hat{\bar{\mathbf{B}}}^{k_n} - \hat{\bar{\mathbf{B}}}\|$, for $n = 1, 2, \ldots$, and the sequences $\{\|\hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} - \hat{\bar{\mathbf{B}}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)\|\}$ and $\{\|\hat{\bar{\mathbf{B}}}^{k_n} - \hat{\bar{\mathbf{B}}}\|\}$ converge to 0. Then, the sequence $\{q_{\tau\rho}^{\Gamma^{k_n}}\}$ converges to q. We have that $u_{\tau\rho}(\mathbf{x}(\tau\rho, \hat{\mathbf{b}}^{\Gamma^{k_n}}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{k_n}}))) \geq u_{\tau\rho}(\mathbf{x}(\tau\rho, \tilde{b}(\tau\rho), p(\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho))))$ as $\hat{\mathbf{b}}^{\Gamma^{k_n}}$ is an atom-type-symmetric Cournot-Nash equilibrium of Γ^{k_n}, for $n = 1, 2, \ldots$. Then, we have that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{k_n}(\tau), p^{k_n})) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau\rho), q_{\tau\rho}^{\Gamma^{k_n}}))$ as $u_{\tau\rho}(\cdot) = u_\tau(\cdot)$, $\hat{\mathbf{b}}^{\Gamma^{k_n}}(\tau\rho) = \hat{\mathbf{b}}^{k_n}(\tau)$, $p(\mathbf{b}^{\Gamma^{k_n}}) = \hat{p}^{k_n}$, $p(\hat{\mathbf{b}}^{\Gamma^{k_n}} \setminus \tilde{b}(\tau\rho)) = \beta_{k_n} q_{\tau\rho}^{\Gamma^{k_n}}$, with $\beta_{k_n} > 0$, by Lemma 2 in Sahi and Yao, for $n = 1, 2, \ldots$. But then, it must be that

$$u_\tau(\hat{\mathbf{x}}(\tau)) = u_\tau \mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), \hat{p}) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau\rho), q) = u_\tau(\tilde{x}),$$

as the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$, the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$, the sequence $\{q_{\tau\rho}^{\Gamma^{k_n}}\}$ converges to q, $\tilde{b}(\tau\rho) = \tilde{b}$, $\hat{p} = \theta q$, with $\theta > 0$, by Lemma 2 in Sahi and Yao, and $u_\tau(\cdot)$ is continuous, by Assumption 2, a contradiction.

Case 2. $\tau \in T_0$. Let $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ be a subsequence of the sequence $\{\hat{\mathbf{b}}^{k_n}(\tau)\}$ which converges to $\hat{\mathbf{b}}(\tau)$. Moreover, let $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau)$ be a strategy selection obtained by replacing $\hat{\mathbf{b}}^{h_{k_n}}(\tau)$ in $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}$ with $\tilde{b}$, for $n = 1, 2, \ldots$. We have that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}(\tau), p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}))) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau), p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau))))$ as $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}$ is an atom-type-symmetric Cournot-Nash equilibrium of $\Gamma^{h_{k_n}}$, for $n = 1, 2, \ldots$. Then, we have that $u_\tau(\mathbf{x}(\tau, \hat{\mathbf{b}}^{h_{k_n}}(\tau), \hat{p}^{h_{k_n}})) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau), \hat{p}^{h_{k_n}}))$ as $\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}(\tau) = \hat{\mathbf{b}}^{h_{k_n}}(\tau)$, $p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}) = \hat{p}^{h_{k_n}}$, and $p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}} \setminus \tilde{b}(\tau)) = p(\hat{\mathbf{b}}^{\Gamma^{h_{k_n}}}) = \hat{p}^{h_{k_n}}$, by Lemma 1 in Codognato and Ghosal (2000). But then, it must be that

$$u_\tau(\hat{\mathbf{x}}(\tau)) = u_\tau \mathbf{x}(\tau, \hat{\mathbf{b}}(\tau), \hat{p}) \geq u_\tau(\mathbf{x}(\tau, \tilde{b}(\tau), \hat{p}) = u_\tau(\tilde{x}),$$

as the sequence $\{\hat{\mathbf{b}}^{h_{k_n}}(\tau)\}$ converges to $\hat{\mathbf{b}}(\tau)$, the sequence $\{p^{h_{k_n}}\}$ converges to $\hat{p}$, $\tilde{b}(\tau) = \tilde{b}$, and $u_\tau(\cdot)$ is continuous, by Assumption 2, a contradiction. Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium of $\mathcal{E}$. ■

7 Discussion of the model

In this section, we go deeper into the relationships of the analysis developed in this paper with the previous literature in the same line.

Let us consider first the contribution by Busetto et al. (2017). The fundamental assumptions underlying the results of these authors are Assumptions 2, 3, and the following two further assumptions.

Assumption 1′. $\mathbf{w}(t) > 0$, *for each* $t \in T$.

Assumption 1′ is clearly less restrictive than our Assumption 1

Assumption 4′. *There are at least two traders in* T_1 *for whom* $\mathbf{w}(t) \gg 0$, u_t *is continuously differentiable in* R^l_{++}, *and* $\{x \in R^l_+ : u_t(x) = u_t(\mathbf{w}(t))\} \subset R^l_{++}$

Assumption 4′ was originally introduced by Sahi and Yao (1989) and reformulated for the mixed version of the Shapley window model by Busetto et al. (2011).

To prove their results Busetto et al. (2017) also needed to use the following notion of a δ-positive strategy selection, which was first used by Sahi and Yao (1989): let $\bar{T}_1 \subset T_1$ be a set consisting of two traders in T_1 for whom Assumption 4′ holds; moreover, let $\delta = \min_{t \in \bar{T}_1}\{\frac{1}{l}\min\{\mathbf{w}^1(t), \ldots, \mathbf{w}^l(t)\}\}$. We say that the correspondence $\mathbf{B}^\delta : T \to \mathcal{P}(R^{l^2}_+)$ is a δ-positive strategy correspondence if $\mathbf{B}^\delta(t) = \mathbf{B}(t) \cap \{(b_{ij}) \in R^{l^2}_+ : \sum_{i \notin J}\sum_{j \in J}(b_{ij} + b_{ji}) \geq \delta,\ \text{for each } J \subseteq \{1, \ldots, l\}\}$, for each $t \in \bar{T}_1$ and if $\mathbf{B}^\delta(t) = \mathbf{B}(t)$, for the remaining traders $t \in T$. Moreover, we say that a strategy selection $\mathbf{b}$ is δ-positive if $\mathbf{b}(t) \in \mathbf{B}^\delta(t)$, for each $t \in T$. This notion can be straightforwardly extended to Γ^n noticing that $\mathbf{B}^\delta(tr) = \mathbf{B}^\delta(ts)$, $r, s = 1, \ldots, n$, for each $t \in T_1$. Then, we say that an atom-type-symmetric Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ^n is δ-positive if $\hat{\mathbf{b}}$ is a δ-positive strategy selection.

Under Assumptions 1′, 2, 3, and 4′, Busetto et al. (2017) proved the existence of a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n, in their Theorem 2, and its convergence to a Walras equilibrium through a replication *à la* Cournot, in their Theorem 3. The proofs of these theorems, as already said, crucially rest on their price convergence result, establishing that any convergent sequence of normalized prices corresponding to a sequence of Cournot-Nash equilibria has a convergent subsequence whose limit is a strictly positive normalized price vector. This result in turn exploits Lemma 9 in Sahi and Yao (1989), which is itself essentially based on hypotheses like those stated in Assumption $4'$.

Let us consider now the contribution by Busetto et al. (2018). Their price convergence theorem, presented in Section 3 and used in this paper to show our Theorems 2 and 3, was employed by those authors to prove, in their Theorem 3, a kind of hybrid existence result based on Assumptions 1, 2, 3, and the following variant of Assumption $4'$.

Assumption $4''$. *There are at least two traders in T_1 for whom $\mathbf{w}(t) \gg 0$.*

This assumption is less restrictive than Assumption $4'$, as it removes the restriction that the two atoms with strictly positive endowments also have continuously differentiable utility functions, and indifference curves contained in the strict interior of the commodity space. We present now two theorems which extend Theorem 3 in Busetto et al. (2018). Under Assumptions 1, 2, 3, and $4''$, and using the price convergence result expressed by Theorem 1, they establish the existence of a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n and its convergence to a Walras equilibrium through a replication *à la* Cournot.

Theorem 4. *Under Assumptions 1, 2, 3, and $4''$, there exists a δ-positive atom-type-symmetric Cournot-Nash equilibrium $\hat{\mathbf{b}}$ of Γ^n.*

Proof. It can be proved by adapting the arguments provided by Theorems 1 and 2, Theorem 2 in Busetto et al. (2017), and Theorem 3 in Busetto et al. (2018). ■

Theorem 5. *Under Assumptions 1, 2, 3, and $4''$, let $\{\hat{\mathbf{b}}^n\}$ be a sequence of strategy selections of Γ and let $\{\hat{p}^n\}$ be a sequence of prices such that $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(tr)$, $r = 1, \ldots, n$, for each $t \in T_1$, $\hat{\mathbf{b}}^n(t) = \hat{\mathbf{b}}^{\Gamma^n}(t)$, for each $t \in T_0$, and $\hat{p}^n = p(\hat{\mathbf{b}}^{\Gamma^n})$, where $\hat{\mathbf{b}}^{\Gamma^n}$ is a δ-positive atom-type-symmetric Cournot-Nash equilibrium of Γ^n, for $n = 1, 2, \ldots$. Then,*
(i) there exists a subsequence $\{\hat{\mathbf{b}}^{k_n}\}$ of the sequence $\{\hat{\mathbf{b}}^n\}$, a subsequence $\{\hat{p}^{k_n}\}$ of the sequence $\{\hat{p}^n\}$, a strategy selection $\hat{\mathbf{b}}$ of Γ, and a price vector $\hat{p} \in \Delta \setminus \partial\Delta$, such that $\hat{\mathbf{b}}(t)$ is the limit of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_1$, $\hat{\mathbf{b}}(t)$ is a limit point of the sequence $\{\hat{\mathbf{b}}^{k_n}(t)\}$, for each $t \in T_0$, the sequence $\{\hat{\bar{\mathbf{B}}}^{k_n}\}$ converges to $\hat{\bar{\mathbf{B}}}$, and the sequence $\{\hat{p}^{k_n}\}$ converges to $\hat{p}$;

(ii) $\hat{\mathbf{x}}(t)$ *is the limit of the sequence* $\{\hat{\mathbf{x}}^{k_n}(t)\}$*, for each* $t \in T_1$*, and* $\hat{\mathbf{x}}(t)$ *is a limit point of the sequence* $\{\hat{\mathbf{x}}^{k_n}(t)\}$*, for each* $t \in T_0$*, where* $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$ *for each* $t \in T$*, and* $\hat{\mathbf{x}}^{k_n}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^{k_n}(t), \hat{p}^{k_n})$*, for each* $t \in T$*, and for* $n = 1, 2, \ldots$*;*
(iii) the pair $(\hat{p}, \hat{\mathbf{x}})$ *is a Walras equilibrium of* $\mathcal{E}$*.*

Proof. It can be proved by adapting the arguments provided by Theorems 1 and 3, and Theorem 3 in Busetto et al. (2017). ■

Codognato and Ghosal (2000) reformulated the Shapley window model, first proposed by Sahi and Yao (1989) for the case of an exchange economy with a finite set of traders, in the context of an exchange economy with an atomless continuum of traders. In this framework, they showed an equivalence result *à la* Aumann (1964) between the set of the Cournot-Nash allocations of the Shapley window model and the set of the Walras allocations of the underlying exchange economy. Since the mixed measure space we are using in this paper may contain countably infinite atoms, the question can be raised whether an equivalence result can be obtained also in this case. We repropose here an example provided by Busetto et al. (2017), which gives a negative answer to the question.

Example 1. *Consider an exchange economy* $\mathcal{E}$*, satisfying Assumptions 1, 2, 3, and 4, where* $l = 2$*,* T_1 *contains countably infinite atoms, there is an atom* $\tau \in T_1$ *such that* $\mathbf{w}^1(\tau) = 0$*,* $\mathbf{w}^2(\tau) > 0$*,* $u_\tau(x) = \sum_{i=1}^2 v_\tau^i(x^i)$*, for each* $x \in R_+^2$*,* $v_\tau^i(x^i)$ *is differentiable, and* $\frac{dv_\tau^i(0)}{dx^i} = +\infty$*,* $i = 1, 2$*. If* $\hat{\mathbf{b}}$ *is a Cournot-Nash equilibrium of* Γ*, then the Cournot-Nash allocation* $\hat{\mathbf{x}}$ *such that* $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$*, for each* $t \in T$*, is not a Walras allocation of* $\mathcal{E}$*.*

Proof. See the proof of the Example in Busetto et al. (2017). ■

While Example 1 proves that the condition that $\mathcal{E}$ contains a countably infinite number of atoms is not sufficient to guarantee that any Cournot-Nash allocation is a Walras allocation, the following example, borrowed from Codognato et al. (2015), can be used to show that this condition is not even necessary.

Example 2. *Consider an exchange economy* $\mathcal{E}$ *satisfying Assumptions 1, 2, 3, and 4, where* $l = 2$*,* $T_1 = \{2\}$*,* $\mu(2) = 1$*,* $\mathbf{w}(2) = (0, 4)$*,* $u_2(x) = \sqrt{x^1} + \frac{1}{30}x^2$*,* $T_0 = [0, 1]$ *is taken with Lebesgue measure,* $\mathbf{w}(t) = (4, 0)$*,* $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$*, for each* $t \in [0, \frac{1}{2}]$*,* $\mathbf{w}(t) = (0, 4)$*,* $u_t(x) = \sqrt{x^1} + \frac{1}{30}x^2$*, for each* $t \in [\frac{1}{2}, 1]$*. Then, there is a unique Walras allocation of* $\mathcal{E}$ *which coincides with the unique Cournot-Nash allocation of* Γ*.*

Proof. The unique Walras equilibrium is the pair $(p^*, \mathbf{x}^*)$, where $(p^{*1}, p^{*2}) = (\frac{\sqrt{21}+3}{2}, 1)$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\frac{8}{\sqrt{21}+3}, 0)$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{8}{\sqrt{21}+5}, 12)$, for each $t \in [0, \frac{1}{2}]$, $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{8}{\sqrt{21}+3}, 0)$, for each $t \in [\frac{1}{2}, 1]$. The strategy selection $\mathbf{b}^*$, where $\mathbf{b}^*_{21}(2) = 4$, $\mathbf{b}^*_{12}(t) = \frac{4\sqrt{21}+12}{\sqrt{21}+5}$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{b}^*_{21}(t) = 4$, for each $t \in [\frac{1}{2}, 1]$, is the unique Cournot-Nash equilibrium and $\mathbf{x}^*(t) = \mathbf{x}(t, \mathbf{b}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Then, the unique Walras allocation is also the unique Cournot-Nash allocation. ■

In the framework of mixed exchange economies, Gabszewicz and Mertens (1971) showed that, if atoms are not "too" big, the core coincides with the set of Walras allocations whereas Shitovitz (1973), in his Theorem B, proved that this result also holds if the atoms are of the same type, i.e., they have the same endowments and preferences.

Okuno et al. (1980) considered a mixed exchange economy with two commodities which are both held by all traders and they showed that, if there are two atoms of the same type who, at a Cournot-Nash equilibrium, demand a positive amount of the two commodities, then the corresponding Cournot-Nash allocation is not a Walras allocation. They contrasted this result with the equivalence between the core and the set of Walras allocations which would hold in this case according to Theorem B in Shitovitz (1973).

Codognato et al. (2015), within the bilateral oligopoly version of the two-commodity mixed exchange economy analysed by Okuno at al. (1980), showed a theorem establishing that, under the assumptions that all traders' utility functions are continuous, strongly monotone, quasi-concave, and measurable, and atoms' utility functions are also differentiable, a necessary and sufficient condition for a Cournot-Nash allocation to be a Walras allocation is that all atoms demand a null amount of one of the two commodities.

Example 2 above satisfies the assumptions used in the main theorem in Gabszewicz and Mertens (1971), in the main theorem in Codognato et al. (2015), and in our limit theorem. This raises the question of the relation between atoms' Walrasian Cournot-Nash strategies and their Walrasian limit.

In the following example, we use the same economy considered in Example 2 to provide a first insight into this issue.

Example 2′. *Consider the exchange economy specified in Example 2. Let* $\hat{\mathbf{b}}^1$ *and* $\hat{\mathbf{b}}$ *be strategy selections as in the statement of Theorem 3. Then,* $\hat{\mathbf{b}}^1 = \hat{\mathbf{b}}$.

Proof. $\hat{\mathbf{b}}^1$ is the unique Cournot-Nash allocation of Γ and the allocation

$\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}^1(t), p(\hat{\mathbf{b}}^1))$, for each $t \in T$, is the unique Walras allocation of $\mathcal{E}$, by Example 2. We also have that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, by Theorem 3. Hence, it must be that $\hat{\mathbf{b}}^1 = \hat{\mathbf{b}}$. ■

This example shows that traders may achieve a Walras allocation at the same Cournot-Nash equilibrium in a finite and an asymptotic economy, i.e., by keeping their strategic power even when atoms become asymptotically negligible.

Finally, we propose a proposition which provides sufficient conditions under which the result obtained by Okuno et al. (1980) we have mentioned above also holds in the bilateral oligopoly version of their model: under the assumptions made both by Busetto et al. (2011) and Busetto et al. (2017), a δ-positive Cournot-Nash allocation is never a Walras allocation.

Proposition. *Consider an exchange economy $\mathcal{E}$, satisfying Assumptions $1'$, 2, 3, and $4'$, where $l = 2$. Let $\hat{\mathbf{b}}$ be a δ-positive Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is not a Walras equilibrium.*

Proof. Let $\hat{\mathbf{b}}$ be a δ-positive Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Suppose that the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium. Consider a trader $\tau \in \bar{T}_1$. It must be that $\hat{\mathbf{x}}(\tau) \gg 0$ as $u_\tau(\hat{\mathbf{x}}(\tau)) \geq u_\tau(\mathbf{w}(\tau))$ and $\tau \in \bar{T}_1$, by Assumption $4'$. Moreover, we have that $\hat{\mathbf{b}}_{ij}(\tau) > 0$ for some i, j with $i \neq j$ as $\hat{\mathbf{b}}$ is a δ-positive Cournot-Nash equilibrium. Suppose, without loss of generality, that $\hat{\mathbf{b}}_{12}(\tau) > 0$. At a Cournot-Nash equilibrium, for the atom τ, the marginal rate of substitution must be equal to the marginal rate at which he can trade off commodity 1 for commodity 2 (see Okuno et al. (1980)). Moreover, at a Walras equilibrium, the marginal rate of substitution must be equal to the relative price of commodity 1 in terms of commodity 2. Combining these two conditions, we obtain

$$\frac{dx^2}{dx^1} = -\frac{\hat{p}^1}{\hat{p}^2} \frac{\bar{\hat{\mathbf{b}}}_{12} - \hat{\mathbf{b}}_{12}(\tau)\mu(\tau)}{\bar{\hat{\mathbf{b}}}_{12}} = -\frac{\hat{p}^1}{\hat{p}^2}.$$

Then, it must be that $\hat{\mathbf{b}}_{12}(\tau) = 0$, a contradiction. Hence, the pair $(\hat{p}, \hat{\mathbf{x}})$ is not a Walras equilibrium. ■

8 Conclusion

The main theorem of this paper – Theorem 3 – is a limit result for the mixed version of the Shapley window model proposed by Busetto et al. (2018). It is innovative with respect to previous results in the same line in that it is crucially based on the Walrasian properties of atomless part's behavior, and can be applied to economic structures left uncovered by the limit theorem proved by Busetto et al. (2017). In our theorem, all traders may indeed have corner endowments, and indifference curves which touch the boundary of the consumption set. In particular, it covers the case of bilateral oligopoly with a competitive fringe for each commodity. We leave for further research the problem of proving a limit theorem for a bilateral oligopoly configuration without a competitive fringe, a case which violates Assumption 1 of our Theorem 3.

In the bilateral mixed exchange framework proposed in the seminal paper by Okuno et al. (1980), Codognato et al. (2015) provided an example where the unique Cournot-Nash allocation and the unique Walras allocation of a finite exchange economy satisfying the assumptions of Theorem 3 coincide. In this paper, we have used the same economy as in the example by Codognato et al. (2015) to show that traders keep their strategic power even when atoms become asymptotically negligible, this way confirming the equivalence result also in this case. Moreover, we have proved, through a proposition, that under the assumptions made by Busetto et al. (2011) and Busetto et al. (2017) the Cournot-Nash allocations whose existence they proved are never Walras allocations. We leave for further research an investigation whether the previous results hold beyond the bilateral exchange framework.

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