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Cournotian Duopolistic Firms May Be Walrasian: A Case in the Gabszewicz and Vial Model

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Abstract

In this paper, we present a refreshed version of the original model proposed by Gabszewicz and Vial (1972) and we use their main example to review the main theoretical issues related to the notion of Cournot-Walras equilibrium. We compute, in the Gabszewicz and Vial main example, two different Cournot-Walras equilibria associated with different normalization rules. Moreover, in the same example, we compute a Utility-Cournot-Walras equilibrium as defined by Grodal (1996) and we show that it coincides with the unique Walras equilibrium. Furthermore, using a proposition proved by Grodal (1996), we build a normalization rule with respect to which there is a Cournot-Walras equilibrium that coincides with the Utility-Cournot-Walras equilibrium and hence with the unique Walras equilibrium. To the best of our knowledge, this example provides the first case of Cournotian duopolistic firms being Walrasian in a production economy.
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1 Introduction

Almost fifty years ago, Gabszewicz and Vial (1972) proposed a pathbreaking analysis of oligopolistic interaction *à la* Cournot among firms in a general equilibrium framework, where they introduced the concept of Cournot-Walras equilibrium. In this celebrated contribution, they lucidly recognized the main theoretical issues raised by their own concept: The dependence of the Cournot-Walras equilibrium on the rule chosen to normalize prices and the possible lack of rationality of the maximization of monetary profits as a decision criterion for the firms. These issues, together with some other more technical problems concerning the very existence of a Cournot-Walras equilibrium, were discussed in a conspicuous literature inspired by their seminal article, which is summarized in several surveys (see Mas-Colell (1982), Hart (1985), Bonanno (1990), among others).

In this paper, we present a refreshed version of the original model proposed by Gabszewicz and Vial (1972) and provide a systematic treatment of the two fundamental issues mentioned above, concerning price normalization and firms' rationality criteria. On the basis of our analysis, we are able to show a strong result: Cournotian duopolistic firms may be Walrasian.

In our analysis, we largely borrow from Grodal (1996): First of all, we use her notion of a normalized price function - i.e., a function that results from the composition of a normalization rule and a price selection - to re-define the very concept of Cournot-Walras equilibrium.

Gabszewicz and Vial (1972) did not explicitly specify a normalization rule in their general model. Nevertheless, in their main example (see p. 385) they introduced a specific rule, which normalizes the prices of an exchange economy using the feasible production plans determining its intermediate initial endowments. They used this normalization rule to compute a Cournot-Walras equilibrium for a two consumers, two firms, and two goods specification of their model.

They employed the same rule also in a second example (see p. 398) to show that a Cournot-Walras equilibrium computed according to it does not coincide with another Cournot-Walras equilibrium computed according to a more standard normalization rule based on one good used as the numeraire. This second example has the drawback of considering a production economy with two firms, two goods but only one consumer, thereby disposing of the Walrasian flavor of the Gabszewicz and Vial model.

Here, in our definition of a Cournot-Walras equilibrium, we base the construction of the notion of a normalized price function on the definition of a type of normaliziation rules which takes inspiration from the one used by Gabszewicz e Vial in their examples and encompasses it. At the same time, this type of normalization rules generalizes a different type of normalization rules, formally introduced by Grodal (1996), which depend only on prices, allowing them to depend also on the quantities produced by firms. We call *à la* Gabszewicz and Vial those normalization rules that satisfy the requirements of our generalization, while we call *à la* Grodal those normalization rules which belong to the type introduced by this author. In Section 2, we formally establish the general relationship between the two types of rules.

After re-considering the notion of a Cournot-Walras equilibrium and proposing a slightly amended version of Gabszewicz and Vial's main example, we compute, in the same basic structure, a Cournot-Walras equilibrium, using a normalization rule *à la* Grodal, that maps prices from the unit simplex into itself. Since this equilibrium differs from that computed by Gabszewicz and Vial on the basis of their normalization rule, the result represents a first explicit proof of the dependence of the original Gabszewicz and Vial's Cournot-Walras equilibrium on the normalization rule, which avoids the shortcomings exhibited by the second example provided by those authors.

Gabszewicz and Vial (1972) also reported an argument proposed in verbal terms by a referee of their original paper concerning firms' rationality criteria (see p. 395). We develop here that argument, providing a result that, to the best of our knowledge, can be considered as a first formal counterexample to the idea, present in the literature, that profit maximization is an arbitrary rationality criterion for firms' owners in a general equilibrium model with oligopolistic interaction *à la* Cournot (see, among others, Grodal (1996)).

Finally, still in the structure of Gabszewicz and Vial's main example, we compute a Utility-Cournot-Walras equilibrium - a notion introduced by Grodal (1996) - and we show that it coincides with the unique Walras equilibrium of the economy. Moreover, using a proposition mutuated once again by Grodal (1996), we build a normalization rule *à la* Grodal with respect to which there is a Cournot-Walras equilibrium that coincides with the Utility-Cournot-Walras equilibrium and hence with the unique Walras equilibrium.

Codognato et al. (2015) and Busetto et al. (2020) exhibited some cases in which atomic Cournotian traders may be Walrasian in pure exchange economies but, as far as we know, our equivalence result, obtained in the basic setup of the Gabszewicz and Vial example, provides the first case of Cournotian duopolistic firms being Walrasian in a production economy.

We review the critical issues raised by Gabszewicz and Vial's seminal analysis in Section 4 within the same structure of a production economy studied by these authors in their main example. Being aware of those theoretical problems, Gabszewicz and Vial (1972) themselves anticipated: "[...] Some readers could accordingly be tempted to reject our theory as a whole; but they should be aware that they would simultaneously reject the whole theory of imperfect competition in partial analysis" (see p. 400).

Nevertheless, partial equilibrium analysis *à la* Cournot is not embodied into a monolithic theory, but it consists of a variety of models designed to capture relevant features of the markets under consideration. Consequently, it seems to us that the main lesson we can draw from our review of the fundamental theoretical questions raised by the general equilibrium analysis *à la* Cournot introduced by Gabszewicz and Vial (1972) is represented by the fact that they showed its limits as a monolithic theory. These limits have been emphasized during a fifty-year long debate which led to a theoretical impasse, preventing the development of a variety of models with production aimed at grasping different configurations of market interrelations which could be considered as a general equilibrium counterpart of the variety of Cournotian models in a partial equilibrium analysis. Rather, the Cournot-Walras approach has shown its major results in the context of pure-exchange economies, where the critical problems listed above are radically removed (see, for instance, Codognato and Gabszewicz (1991)).

In the last section of this work, we shall have a look at some promising very recent developments of the theory of oligopoly *à la* Cournot in a general equilibrium analysis, which could overcome the impasse of this theory for production economies.

The paper is organized as follows. In Section 2, we introduce the mathematical model and we define the notion of Cournot-Walras equilibrium inroducing the notion of normalized price function. In Section 3, we compute two different Cournot-Walras equilibria in the structure of Gabszewicz and Vial's main example. In Section 4, we discuss, through further results, the main issues related to the notion of Cournot-Walras equilibrium for economies with production. In Section 5, we draw some conclusions and we suggest some further lines of research.

2 Mathematical model

We present here a refreshed version of the mathematical model proposed by Gabszewicz and Vial (1972), where we explicitly specify the notion of a normalization rule, generalizing that proposed by Grodal (1996), to define the concept of Cournot-Walras equilibrium.

We consider a production economy with n consumers i, $i = 1, \ldots, n$, m firms j, $j = 1, \ldots, m$, and l consumption goods, $h = 1, \ldots, l$.

Each consumer $i = 1, \ldots, n$ is characterized by a consumption set R^l_+, an initial endowment vector $\omega_i \in R^l_+$, with $\omega_i \gg 0$, a share θ_{ij} in the production of firm j, such that $\sum_{i=1}^n \theta_{ij} = 1$, for each firm $j = 1, \ldots m$, and a rational, continuous, strongly monotone, and strictly convex preference relation $\precsim_i$, defined on R^l_+. A consumption bundle of consumer i is a vector $x_i \in R^l_+$.

Each firm $j = 1, \ldots, m$ is characterized by a compact and convex production set $G_j \subset R^l_+$. A feasible production plan of firm j is a vector $y_j \in G_j$.

A price vector is a vector $p \in \Delta$, where Δ is the unit simplex.

An allocation is a n-tuple of consumption bundles $(x_1, \ldots, x_n)$ and a m-tuple of feasible production plans $(y_1, \ldots, y_m)$ such that $\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i + \sum_{j=1}^m y_j$.

Given feasible production plans $(y_1, \ldots, y_m)$, the intermediate endowment of consumer i is $\omega_i + \sum_{j=1}^m \theta_{ij} y_j$.

Given feasible production plans $(y_1, \ldots, y_m)$, an equilibrium allocation relative to $(y_1, \ldots, y_m)$ is a n-tuple of consumption bundles $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^n x_i = \sum_{i=1}^n (\omega_i + \sum_{j=1}^m \theta_{ij} y_j)$.

A n-tuple of consumption bundles $(x_1, \ldots, x_n)$ and a m-tuple of feasible production plans $(y_1, \ldots, y_m)$ such that $(x_1, \ldots, x_n)$ is an equilibrium allocation relative to $(y_1, \ldots, y_m)$ is an allocation as $\sum_{i=1}^n x_i = \sum_{i=1}^n (\omega_i + \sum_{j=1}^m \theta_{ij} y_j) = \sum_{i=1}^n \omega_i + \sum_{j=1}^m \sum_{i=1}^n \theta_{ij} y_j = \sum_{i=1}^n \omega_i + \sum_{j=1}^m y_j$.

Given feasible production plans $(y_1, \ldots, y_m)$, a Walras equilibrium relative to $(y_1, \ldots, y_m)$ is a pair $(p, (x_1, \ldots, x_n))$ consisting of a price vector $p \in \Delta$ and an equilibrium allocation $(x_1, \ldots, x_n)$ relative to $(y_1, \ldots, y_m)$ such that $x_i \succsim_i x'_i$, for each x'_i such that $px'_i \leq p\omega_i + p\sum_{j=1}^m \theta_{ij} y_j$, for each consumer $i = 1, \ldots, n$.

Given feasible production plans $(y_1, \ldots, y_m)$, there exists a Walras equilibrium $(p, (x_1, \ldots, x_n))$ relative to $(y_1, \ldots, y_m)$ as $\precsim_i$ is rational, continuous, strongly monotone, and strictly convex, for each consumer $i = 1, \ldots, n$.

A price correspondence is a correspondence π defined on $\prod_{j=1}^m G_j$ with values in Δ such that, for all feasible production plans $(y_1, \ldots, y_m)$, $(p, (x_1, \ldots$

x_n)) is a Walras equilibrium relative to $(y_1, \ldots, y_m)$, for some $p \in \pi(y_1, \ldots, y_m)$ and for some equilibrium allocation $(x_1, \ldots, x_n)$.

A price selection is a function p defined on $\prod_{j=1}^m G_j$ with values in Δ such that $p(y_1, \ldots, y_m) \in \pi(y_1, \ldots, y_m)$, for all feasible production plans $(y_1, \ldots, y_m)$.

A normalization rule *à la* Gabszewicz and Vial is a function α defined on $\prod_{j=1}^m G_j \times \Delta$ with values in $R^l_+ \setminus \{0\}$ such that $\alpha(y_1, \ldots, y_m, p) = \sum_{h=1}^l \alpha_h(y_1, \ldots, y_m, p)p$, for all feasible production plans $(y_1, \ldots, y_m)$ and for each $p \in \Delta$.

A normalization rule *à la* Gabszewicz and Vial is a normalization rule *à la* Grodal if and only if $\alpha(y_1, \ldots, y_m, p) = \alpha(y'_1, \ldots, y'_m, p)$, for all feasible production plans $(y_1, \ldots, y_m)$ and $(y'_1, \ldots, y'_m)$ and for each $p \in \Delta$ (see Grodal (1996)).

Given a price selection p and a normalization rule *à la* Gabszewicz and Vial α, a normalized price function is a function p^α defined on $\prod_{j=1}^m G_j$ with values in $R^l_+ \setminus \{0\}$ such that $p^\alpha(y_1, \ldots, y_m) = \alpha(y_1, \ldots, y_m, p(y_1, \ldots, y_m))$, for all feasible production plans $(y_1, \ldots, y_m)$.

Given a normalized price function p^α, the profit function of firm j is the function $p^\alpha(y_1, \ldots, y_m)y_j$, for all feasible production plans $(y_1, \ldots, y_m)$.

Given a normalized price function p^α, a m-tuple of feasible production plans $(y_1^*, \ldots, y_m^*)$ is a Cournot equilibrium for p^α if

$$p^\alpha(y_1^*, \ldots, y_j^* \ldots, y_m^*)y_j^* \geq p^\alpha(y_1^*, \ldots, y_j \ldots, y_m^*)y_j,$$

for each $y_j \in G_j$ and for each firm $j = 1, \ldots, m$.

A Cournot-Walras equilibrium is a triplet $(p^\alpha, (x_1^*, \ldots, x_n^*), (y_1^*, \ldots, y_m^*))$ consisting of a normalized price function p^α, a m-tuple of feasible production plans $(y_1^*, \ldots, y_m^*)$, and an equilibrium allocation $(x_1^*, \ldots, x_n^*)$ relative to $(y_1^* \ldots, y_m^*)$ such that the pair $(p^\alpha(y_1^*, \ldots, y_m^*)), (x_1^*, \ldots, x_n^*))$ is a Walras equilibrium relative to $(y_1^*, \ldots, y_m^*)$ and $(y_1^*, \ldots, y_m^*)$ is a Cournot equilibrium for p^α.

A Walras equilibrium of the production economy is a triplet $(\hat{p}, (\hat{x}_1, \ldots, \hat{x}_n), (\hat{y}_1, \ldots, \hat{y}_m))$ consisting of a price vector $\hat{p} \in \Delta$, a m-tuple of feasible production plans $(\hat{y}_1, \ldots, \hat{y}_m)$, and an equilibrium allocation $(\hat{x}_1, \ldots, \hat{x}_n)$ relative to $(\hat{y}_1, \ldots, \hat{y}_m)$ such that the pair $(\hat{p}, (\hat{x}_1, \ldots, \hat{x}_n))$ is a Walras equilibrium relative to $(\hat{y}_1, \ldots, \hat{y}_m)$ and $\hat{p}y_j$ achieves its maximum on G_j in $\hat{y}_j$, for each firm $j = 1, \ldots, m$.

3 Cournot-Walras equilibrium and normalization rules

In order to illustrate the fundamental concepts introduced in their paper, Gabszewicz and Vial (1972) considered a first example - the main one of their paper - which constitutes a particularization of the model of a production economy introduced in the previous section. In this example, they used a specific normalization rule belonging to the type we have called *à la* Gabszewicz and Vial.

Here, we present a more articulated example, in which we first re-propose a slightly amended version of their result. Then, within the same structure of a production economy introduced by those authors, we compute a different Cournot-Walras equilibrium, using a type of normalization rule *à la* Grodal. This result provides a first explicit proof of the dependence of the original Gabszewicz and Vial's Cournot-Walras equilibrium on the normalization rule. As anticipated in the Introduction, Gabszewicz and Vial's original paper contains a second example in which a different Cournot-Walras equilibrium is associated to a different normalization rule. Neverthless, it considers a simple production economy with a unique consumer, which cannot be adequately compared with the general equilibrium setup used in their main example. The Example below aims at overcoming these limitations.

Example. *Consider a production economy, where* $i = 2$, $j = 2$, $l = 2$, $\omega_1 = (0,0)$, $\omega_2 = (0,0)$, $\theta_{11} = 1$, $\theta_{12} = 0$, $\theta_{21} = 0$, $\theta_{22} = 1$, $\succsim_1$ *is represented by the utility function* $u_1(x_{11}, x_{21}) = x_{11}^{\frac{1}{4}} x_{21}^{\frac{3}{4}}$, $\succsim_2$ *is represented by the utility function* $u_2(x_{12}, x_{22}) = x_{12}^{\frac{3}{4}} x_{22}^{\frac{1}{4}}$, $G_1 = \{y_1 = (y_{11}, y_{21}) : 0 \leq y_{11} \leq 2, 0 \leq y_{21} \leq 8, 2y_{11} + y_{21} \leq 10\}$, $G_2 = \{y_2 = (y_{12}, y_{22}) : 0 \leq y_{12} \leq 8, 0 \leq y_{22} \leq 2, y_{12} + 2y_{22} \leq 10\}$. *Moreover, consider the functions* $\beta(y_1, y_2, p) = p$ *and* $\gamma(y_1, y_2, p) = \frac{3y_{11}+y_{12}+y_{21}+3y_{22}}{D} p$, *where* $D = (y_{21} + 3y_{22})(y_{11} + y_{12}) + (3y_{11} + y_{12})(y_{21} + y_{22})$, *for all feasible production plans* (y_1, y_2) *and for each* $p \in \Delta$. *Then, the triplet* $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$, *where* $\hat{p} = (\frac{1}{2}, \frac{1}{2})$, $(\hat{x}_1, \hat{x}_2) = ((\frac{9}{4}, \frac{27}{4}), (\frac{27}{4}, \frac{9}{4}))$, $(\hat{y}_1, \hat{y}_2) = ((1, 8), (8, 1))$, *is the unique Walras equilibrium of the production economy; the function* p, *where* $p(y_1, y_2) = (\frac{y_{21}+3y_{22}}{3y_{11}+y_{12}+y_{21}+3y_{22}}, \frac{3y_{11}+y_{12}}{3y_{11}+y_{12}+y_{21}+3y_{22}})$, *for all feasible production plans* (y_1, y_2), *is the unique price selection;* β *is a normalization rule à la Grodal and the triplet* $(p^\beta, (x_1^*, x_2^*), (y_1^*, y_2^*))$, *where* $p^\beta(y_1, y_2) = (\frac{y_{21}+3y_{22}}{3y_{11}+y_{12}+y_{21}+3y_{22}}, \frac{3y_{11}+y_{12}}{3y_{11}+y_{12}+y_{21}+3y_{22}})$, *for all feasible production plans* (y_1, y_2), $(x_1^*, x_2^*) = ((\frac{35}{17}, \frac{105}{17}), (\frac{105}{17}, \frac{35}{17}))$, $(y_1^*, y_2^*) = ((\frac{30}{17}, \frac{110}{17}), (\frac{110}{17}, \frac{30}{17}))$, *is a Cournot-Walras equilibrium; γ is a normalization rule à la Gabszewicz and Vial and the triplet $(p^\gamma, (x_1^{**}, x_2^{**}), (y_1^{**}, y_2^{**}))$, where $p^\gamma(y_1, y_2) = (\frac{y_{21}+3y_{22}}{D}, \frac{3y_{11}+y_{12}}{D})$, for all feasible production plans (y_1, y_2), $(x_1^{**}, x_2^{**}) = ((2,6),(6,2))$, $(y_1^{**}, y_2^{**}) = ((2,6),(6,2))$, is a Cournot-Walras equilibrium.*

Proof. Given feasible production plans (y_1, y_2), the demand function of consumer 1 is

$$x_1(p, py_1) = \left(\frac{p_1y_{11}+p_2y_{21}}{4p_1}, \frac{3(p_1y_{11}+p_2y_{21})}{4p_2}\right)$$

and the demand function of consumer 2 is

$$x_2(p, py_2) = \left(\frac{3(p_1y_{12}+p_2y_{22})}{4p_1}, \frac{p_1y_{12}+p_2y_{22}}{4p_2}\right).$$

The triplet $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$, where $\hat{p} = (\frac{1}{2}, \frac{1}{2})$, $(\hat{x}_1, \hat{x}_2) = ((\frac{9}{4}, \frac{27}{4}), (\frac{27}{4}, \frac{9}{4}))$, $(\hat{y}_1, \hat{y}_2) = ((1,8),(8,1))$, is a Walras equilibrium of the production economy as

$$\hat{x}_{11}+\hat{x}_{12} = x_{11}(\hat{p}, \hat{p}\hat{y}_1) + x_{12}(\hat{p}, \hat{p}\hat{y}_2) = \frac{9}{4} + \frac{27}{4} = 1+8 = \hat{y}_{11}+\hat{y}_{12},$$

$$\hat{x}_{21}+\hat{x}_{22} = x_{21}(\hat{p}, \hat{p}\hat{y}_1) + x_{22}(\hat{p}, \hat{p}\hat{y}_2) = \frac{27}{4} + \frac{9}{4} = 8+1 = \hat{y}_{21}+\hat{y}_{22},$$

$\hat{p}\hat{y}_1 = \frac{1}{2}y_{11} + \frac{1}{2}y_{21}$ achieves its maximum on G_1 in $\hat{y}_1 = (1,8)$, and $\hat{p}\hat{y}_2 = \frac{1}{2}y_{12} + \frac{1}{2}y_{22}$ achieves its maximum on G_2 in $\hat{y}_2 = (8,1)$. We now show that the triplet $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$ is the unique Walras equilibrium of the production economy. Suppose that there exists a triplet $(\tilde{p}, (\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)) \neq (\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$ which is a Walras equilibrium of the production economy. Suppose that $\frac{\tilde{p}_1}{\tilde{p}_2} \geq 2$. Then, it is straightforward to verify that $\tilde{y}_1 = (t2+(1-t), t6+(1-t)8)$, for some $t \in [0,1]$ and $\tilde{y}_2 = (8,1)$. But then, we have that

$$\tilde{x}_{11}+\tilde{x}_{12} < \tilde{y}_{11}+\tilde{y}_{12},$$

a contradiction. Suppose that $\frac{\tilde{p}_1}{\tilde{p}_2} \leq \frac{1}{2}$. Then, it is straightforward to verify that $\tilde{y}_1 = (1,8)$ and $\tilde{y}_2 = (t6+(1-t)8, t2+(1-t))$, for some $t \in [0,1]$. But then, we have that

$$\tilde{x}_{11}+\tilde{x}_{12} > \tilde{y}_{11}+\tilde{y}_{12},$$

a contradiction. Therefore, we must have that $\frac{1}{2} < \frac{\tilde{p}_1}{\tilde{p}_2} < 2$. Then, it is immediate to check that $\tilde{y}_1 = (1,8)$ and $\tilde{y}_2 = (8,1)$ and that $\tilde{p} = (\frac{1}{2}, \frac{1}{2})$ is the only solution to the equation

$$x_{11}(\tilde{p}, \tilde{p}\tilde{y}_1) + x_{12}(\tilde{p}, \tilde{p}\tilde{y}_2) = 9 = x_{21}(\tilde{p}, \tilde{p}\tilde{y}_1) + x_{22}(\tilde{p}, \tilde{p}\tilde{y}_2).$$

But then, we have that $(\tilde{p}, (\tilde{x}_1, \tilde{x}_2), (\tilde{y}_1, \tilde{y}_2)) = (\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$, a contradiction. Therefore, the triplet $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$ is the unique Walras equilibrium of the production economy. The function p, where $p(y_1, y_2) = (\frac{y_{21}+3y_{22}}{3y_{11}+y_{12}+y_{21}+3y_{22}}, \frac{3y_{11}+y_{12}}{3y_{11}+y_{12}+y_{21}+3y_{22}})$, is the unique price selection as $p(y_1, y_2)$ is the unique solution to the system of equations

$$x_{11}(p, py_1) + x_{12}(p, py_2) = y_{11} + y_{12}$$

and

$$p_1 + p_2 = 1,$$

for all feasible production plans (y_1, y_2). Consider the function $\beta(y_1, y_2, p) = p$. β is a normalization rule *à la* Gabszewicz and Vial as $\beta(y_1, y_2, p) = \sum_{h=1}^{2} \beta_h(y_1, y_2, p)p$, for all feasible production plans (y_1, y_2) and for each $p \in \Delta$. Moreover, it is a normalization rule *à la* Grodal as $\beta(y_1, y_2, p) = \beta(y_1', y_2', p)$, for all feasible production plans (y_1, y_2) and (y_1', y_2') and for each $p \in \Delta$. Consider the triplet $(p^\beta, (x_1^*, x_2^*), (y_1^*, y_2^*))$, where $p^\beta(y_1, y_2) = (\frac{y_{21}+3y_{22}}{3y_{11}+y_{12}+y_{21}+3y_{22}}, \frac{3y_{11}+y_{12}}{3y_{11}+y_{12}+y_{21}+3y_{22}})$, for all feasible production plans (y_1, y_2), $(x_1^*, x_2^*) = ((\frac{35}{17}, \frac{105}{17}), (\frac{105}{17}, \frac{35}{17}))$, $(y_1^*, y_2^*) = ((\frac{30}{17}, \frac{110}{17}), (\frac{110}{17}, \frac{30}{17}))$. p^β is a normalized price function as

$$p^\beta(y_1, y_2) = \beta(y_1, y_2, p(y_1, y_2)) = p(y_1, y_2),$$

for all feasible production plans (y_1, y_2). Let $p^{\beta*} = (p^\beta(y_1^*, y_2^*)) = (\frac{1}{2}, \frac{1}{2})$. The pair $(p^{\beta*}, (x_1^*, x_2^*))$ is a Walras equilibrium relative to (y_1^*, y_2^*) as

$$x_{11}^* + x_{12}^* = x_{11}(p^{\beta*}, p^{\beta*}y_1^*) + x_{12}(p^{\beta*}, p^{\beta*}y_2^*) = \frac{35}{17} + \frac{105}{17} = \frac{30}{17} + \frac{110}{17} = y_{11}^* + y_{12}^*$$

and

$$x_{21}^* + x_{22}^* = x_{21}(p^{\beta*}, p^{\beta*}y_1^*) + x_{22}(p^{\beta*}, p^{\beta*}y_2^*) = \frac{105}{17} + \frac{35}{17} = \frac{110}{17} + \frac{30}{17} = y_{21}^* + y_{22}^*.$$

The profit function of firm 1, given the feasible production plan of firm 2, y_2^*, is

$$p^\beta(y_1, y_2^*)y_1 = \frac{68y_{11}y_{21} + 90y_{11} + 110y_{21}}{51y_{11} + 17y_{21} + 200}.$$

Let f_1^β denote the extension of this profit function to R_+^2. f_1^β is strictly increasing in y_{11} and y_{21} as it is straightforward to verify that $\frac{\partial f_1^\beta(y_1, y_2^*)}{\partial y_{11}} > 0$

and $\frac{\partial f_1^\beta(y_1,y_2^*)}{\partial y_{21}} > 0$, for each $y_1 \in R_+^2$. Moreover, it is also possible to verify, through some more cumbersome computations, that

$$-\left(\frac{\partial f_1^\beta(y_1,\check{y}_2)}{\partial y_{11}}\right)^2 \frac{\partial^2 \bar{v}_1(y_1,\check{y}_2)}{\partial y_{21}^2} - \left(\frac{\partial f_1^\beta(y_1,\check{y}_2)}{\partial y_{21}}\right)^2 \frac{\partial^2 f_1^\beta(y_1,\check{y}_2)}{\partial y_{11}^2}$$
$$+2\frac{\partial f_1^\beta(y_1,\check{y}_2)}{\partial y_{11}} \frac{\partial f_1^\beta(y_1,\check{y}_2)}{\partial y_{21}} \frac{\partial^2 f_1^\beta(y_1,\check{y}_2)}{\partial y_{11}\partial y_{12}} > 0,$$

for each $y_1 \in R_+^2$. Then, the function f_1^β is strictly quasi-concave on G_1. At $(y_{11}^*, y_{21}^*) = (\frac{30}{17}, \frac{110}{17})$, $\lambda_1^* = \lambda_2^* = 0$, and $\lambda_3^* = \frac{2}{5}$, the Kuhn-Tucker conditions for the maximization of the function f_1^β on G_1, which reduce to

$$y_{11}(\frac{\partial f_1^\beta(y_1,y_2^*)}{\partial y_{11}} - \lambda_1 - 2\lambda_3) = 0,$$

$$y_{21}(\frac{\partial f_1^\beta(y_1,y_2^*)}{\partial y_{21}} - \lambda_2 - \lambda_3) = 0,$$
$$\lambda_1(y_{11} - 2) = 0,$$
$$\lambda_2(y_{21} - 8) = 0,$$
$$\lambda_3(2y_{11} + y_{21} - 10) = 0,$$

are satisfied as $\frac{\partial f_1^\beta(y_1^*,y_2^*)}{\partial y_{11}} = \frac{4}{5}$ and $\frac{\partial f_1(^\beta y_1^*,y_2^*)}{\partial y_{21}} = \frac{2}{5}$. Then, (y_{11}^*, y_{21}^*) is the unique feasible production plan which maximizes f_1^β on G_1 as f_1 is strictly quasi-concave. The profit function of firm 2, given the feasible production plan of firm 1, y_1^*, is

$$p^\beta(y_1^*, y_2)y_1 = \frac{68y_{12}y_{22} + 90y_{22} + 110y_{12}}{51y_{22} + 17y_{12} + 200}.$$

Let f_2^β denote the extension of this profit function to R_+^2. Then, by using, *mutatis mutandis*, the previous argument, it is straightforward to verify that (y_{12}^*, y_{22}^*) is the unique feasible production plan which maximizes f_2^β on G_2. Therefore, the triplet $(p^\beta, (x_1^*, x_2^*), (y_1^*, y_2^*))$ is a Cournot-Walras equilibrium. Consider the function $\gamma(y_1, y_2, p) = \frac{y_{21}+3y_{22}+3y_{11}+y_{12}}{D}p$. γ is a normalization rule *à la* Gabszewicz and Vial as $\gamma(y_1, y_2, p) = (\sum_{h=1}^2 \frac{y_{21}+3y_{22}+3y_{11}+y_{12}}{D}p_h)p = \sum_{h=1}^2 \gamma_h(y_1, y_2, p)p$, for all feasible production plans (y_1, y_2) and for each $p \in \Delta$. Consider the triplet $(p^\gamma, (x_1^{**}, x_2^{**}), (y_1^{**}, y_2^{**}))$, where $p^\gamma(y_1, y_2) =$

$(\frac{y_{21}+3y_{22}}{D}, \frac{3y_{11}+y_{12}}{D})$, for all feasible production plans (y_1, y_2), $(x_1^{**}, x_2^{**}) = ((2,6),(6,2))$, $(y_1^{**}, y_2^{**}) = ((2,6),(6,2))$. p^γ is a normalized price function as

$$p^\gamma(y_1, y_2) = \gamma(y_1, y_2, p(y_1, y_2)) = \frac{3y_{11} + y_{12} + y_{21} + 3y_{22}}{D} p(y_1, y_2),$$

for all feasible production plans (y_1, y_2). Let $p^{\gamma **} = (p^\gamma(y_1^{**}, y_2^{**})) = (\frac{1}{16}, \frac{1}{16})$. The pair $(p^{\gamma **}, (x_1^{**}, x_2^{**}))$ is a Walras equilibrium relative to (y_1^{**}, y_2^{**}) as

$$x_{11}^{**} + x_{12}^{**} = x_{11}(p^{\gamma **}, p^{\gamma **} y_1^{**}) + x_{12}(p^{\gamma **}, p^{\gamma **} y_2^{**}) = 2 + 6 = y_{11}^{**} + y_{12}^{**}$$

and

$$x_{21}^{**} + x_{22}^{**} = x_{21}(p^{\beta **}, p^{\beta **} y_1^{**}) + x_{22}(p^{\beta **}, p^{\beta **} y_2^{**}) = 6 + 2 = y_{21}^{**} + y_{22}^{**}.$$

The profit function of firm 1, given the feasible production plan of firm 2, y_2^{**}, is

$$p^\gamma(y_1, y_2^{**}) y_1 = \frac{2y_{11}y_{21} + 3y_{11} + 3y_{21}}{2y_{11}y_{21} + 6y_{11} + 6y_{21} + 24}.$$

Let f_1^γ denote the extension of this profit function to R_+^2. f_1^γ is strictly increasing in y_{11} and y_{21} as it is straightforward to verify that $\frac{\partial f_1^\gamma(y_1, y_2^{**})}{\partial y_{11}} > 0$ and $\frac{\partial f_1^\gamma(y_1, y_2^{**})}{\partial y_{21}} > 0$, for each $y_1 \in R_+^2$. Moreover, it is also possible to verify, through some more cumbersome computations, that

$$- \left(\frac{\partial f_1^\gamma(y_1, \check{y}_2)}{\partial y_{11}}\right)^2 \frac{\partial^2 f_1^\gamma(y_1, \check{y}_2)}{\partial y_{21}^2} - \left(\frac{\partial f_1^\gamma(y_1, \check{y}_2)}{\partial y_{21}}\right)^2 \frac{\partial^2 f_1^\gamma(y_1, \check{y}_2)}{\partial y_{11}^2}$$
$$+2 \frac{\partial f_1^\gamma(y_1, \check{y}_2)}{\partial y_{11}} \frac{\partial f_1^\gamma(y_1, \check{y}_2)}{\partial y_{21}} \frac{\partial^2 f_1^\gamma(y_1, \check{y}_2)}{\partial y_{11} \partial y_{12}} > 0,$$

for each $y_1 \in R_+^2$. Then, the function f_1^γ is strictly quasi-concave on G_1. At $(y_{11}^{**}, y_{21}^{**}) = (2,6)$, $\lambda_1^{**} = \lambda_3^{**} = \frac{1}{48}$, and $\lambda_3^{**} = 0$, the Kuhn-Tucker conditions for the maximization of the function f_1^γ on G_1, which reduce to

$$y_{11}(\frac{\partial f_1^\gamma(y_1, y_2^{**})}{\partial y_{11}} - \lambda_1 - 2\lambda_3) = 0,$$

$$y_{21}((\frac{\partial f_1^\gamma(y_1, y_2^{**})}{\partial y_{21}} - \lambda_2 - \lambda_3) = 0,$$

$$\lambda_1(y_{11} - 2) = 0,$$

$$\lambda_2(y_{21} - 8) = 0,$$

$$\lambda_3(2y_{11} + y_{21} - 10) = 0,$$

are satisfied as $\frac{\partial f_1^\gamma(y_1^{**}, y_2^{**})}{\partial y_{11}} = \frac{3}{48}$ and $\frac{\partial f_1^\gamma(y_1^{**}, y_2^{**})}{\partial y_{21}} = \frac{1}{48}$. Then, $(y_{11}^{**}, y_{21}^{**})$ is the unique feasible production plan which maximizes f_1^γ on G_1 as f_1^γ is strictly quasi-concave. The profit function of firm 2, given the feasible production plan of firm 1, y_1^{**}, is

$$p^\beta(y_1^{**}, y_2)y_1 = \frac{2y_{12}y_{22} + 3y_{12} + 3y_{22}}{2y_{12}y_{21} + 6y_{12} + 6y_{22} + 24}$$

Let f_2^γ denote the extension of this profit function to R_+^2. Then, by using, *mutatis mutandis*, the previous argument, it is straightforward to verify that $(y_{12}^{**}, y_{22}^{**})$ is the unique feasible production plan which maximizes f_2^γ on G_2. Therefore, the triplet $(p^\gamma, (x_1^{**}, x_2^{**}), (y_1^{**}, y_2^{**}))$ is a Cournot-Walras equilibrium. ■

4 Discussion of the model and the Example

In Section 3, we have reconsidered the main example proposed by Gabszewicz and Vial (1972), providing more details, amending some minor miscalculations, and adding the computation of a new Cournot-Walras equilibrium under a different normalization rule, *à la* Grodal. We use now the results obtained in the previous section to discuss some crucial points of the theoretical tradition related to the Cournot-Walras equilibrium concept.

A first - minor - technical point concerns the fact that, in the Example, consumers' preferences are represented by Cobb-Douglas utility functions, which are not strongly monotone on the boundary of the consumption set. This minor difficulty can be overcome by assuming that the consumption set is restricted to R_{++}^2.

A second technical point concerns the fact that, in the Example, consumers' endowments are equal to zero. The general model proposed by Gabszewicz and Vial (1972), re-produced in Section 1, requires that each consumer has a strictly positive endowment of each consumption good. Since this model does not consider inputs or intermediary goods - as explicitly recognized by Gabszewicz and Vial (1972) themselves - this assumption guarantees that there exists an exchange economy, i.e., an economy where there is something to be exchanged, relative to all feasible production plans, and, together with the assumptions on consumers' preferences, that the set of its Walras equilibria in nonempty. Actually, the weaker condition $\omega_i \geq 0$, for each consumer $i = 1, \ldots, n$, and $\sum_{i=1}^{n} \omega_i \gg 0$ could be sufficient for the existence of a Walras equilibrium relative to all feasible production plans. However, in the extensions of this model which include inputs this condition is no longer sufficient as the demand for inputs may lead to non-positive intermediate endowments for some consumers. This issue is dealt with by Mas-Colell (1982), Hart (1985), and Bonanno (1990), among others, in their survey articles and is only referred to the problem of nonexistence of a Walras equilibrium of the exchange economies corresponding to these "pathological" intermediate endowments. Nevertheless, a more fundamental problem arises when intermediate endowments are strictly negative for some consumer or null for all consumers: The very existence of an exchange economy. Therefore, in the structure of the Example, an exchange economy relative to null production plans does not exist as there is nothing to be exchanged. Gabszewicz and Vial (1972) themselves were aware of this difficulty which they proposed to overcome "[...] by restricting the strategy set on each firm to the intersection of their respective production set with the strictly positive orthant" (see Footnote 5, p. 386). This restriction can actually be weakened by imposing that the sum of the strategy sets of the two firms is contained in the strictly positive orthant.

The Example in Section 3 compares two different Cournot-Walras equilibria, obtained on the basis of two different types of normalization rules. The first one re-proposes the specific rule introduced by Gabszewicz and Vial (1972) in their main example, which normalizes the prices of an exchange economy using the feasible production plans determining its intermediate initial endowments. This normalization rule, of the type *à la* Gabszewicz and Vial, constitutes a generalization of the other one, *à la* Grodal, depending only on prices. The distinction between these two kinds of normalization rules was recognized by Dierker and Grodal (1986). As is well-known, these authors developed some examples on the non-existence of a Cournot-Walras equilibrium, proposing the following comment: "It should be remarked that we do not allow the normalization to depend on the production plans but only on relative prices (see p. 168). On the other hand, the fact that price normalization also depends on production plans is consistent with what observed by Gabszewicz and Vial (1972) themselves: "[...] the price system only defines a *direction* in the commodity space: this information is not sufficient to specify how the influence that the firms exert on this direction can affect their monetary profits. For the competitive equilibrium concept, one has not to worry about this specification since, *by assumption*, the firms do not exert any influence on the direction of prices. Such a specification is needed, however, if the profit criterion is incorporated into the mode" (see p. 400). As remarked above, Gabszewicz and Vial (1972) did not consider the role of price normalization on profits' maximization in their main example, but they did it using an example of a "degenerate" economy with only one consumer thereby disposing of the Walrasian flavor of their model.

The role of normalization in the determination and the existence of a Cournot-Nash equilibrium was considered by Dieker and Grodal (1986), Böhm (1994), and Ginsburgh (1994), among others. In particular, Ginsburgh (1994) considered an example of a production economy with two goods where each good is produced by a monopolist and he showed that a change in the normalization rules leads to different Cournot-Walras equilibria with different welfare properties.

We study now the welfare properties of the Cournot-Walras equilibria computed in the Section 3.

Example [Continued]**.** *Consider the production economy specified above. Then, the unique Walras equilibrium of the production economy* $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$ *is Pareto optimal, it Pareto dominates the Cournot-Walras equilibrium* $(p^\beta, (x_1^*, x_2^*), (y_1^*, y_2^*))$*, which, in turn, Pareto dominates the Cournot-Walras equilibrium* $(p^\gamma, (x_1^{**}, x_2^{**}), (y_1^{**}, y_2^{**}))$*.*

Proof. The unique Walras equilibrium of the production economy $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$ is Pareto optimal by the first fundamental theorem of welfare economics. It Pareto dominates the Cournot-Walras equilibrium $(p^\beta, (x_1^*, x_2^*), (y_1^*, y_2^*))$ as

$$u_1(\hat{x}_1) = u_1\left(\frac{9}{4}, \frac{27}{4}\right) = \frac{9}{4}(3)^{\frac{3}{4}} > \frac{35}{17}(3)^{\frac{3}{4}} = u_1\left(\frac{35}{17}, \frac{105}{17}\right) = u_1(x_1^*)$$

and

$$u_2(\hat{x}_2) = u_2\left(\frac{27}{4}, \frac{9}{4}\right) = \frac{9}{4}(3)^{\frac{3}{4}} > \frac{35}{17}(3)^{\frac{3}{4}} = u_2\left(\frac{105}{17}, \frac{35}{17}\right) = u_2(x_2^*).$$

The Cournot-Walras equilibrium $(p^\beta, (x_1^*, x_2^*), (y_1^*, y_2^*))$ Pareto dominates the Cournot-Walras equilibrium $(p^\gamma, (x_1^{**}, x_2^{**}), (y_1^{**}, y_2^{**}))$ as

$$u_1(x_1^*) = u_1\left(\frac{35}{17}, \frac{105}{17}\right) = \frac{35}{17}(3)^{\frac{3}{4}} > 2(3)^{\frac{3}{4}} = u_1(2, 6) = u_1(x_1^{**})$$

and

$$u_2(x_2^*) = u_2\left(\frac{105}{17}, \frac{35}{17}\right) = \frac{35}{17}(3)^{\frac{3}{4}} > 2(3)^{\frac{3}{4}} = u_2(6, 2) = u_2(x_2^{**}).$$

■

Therefore, the two Cournot-Walras equilibria computed in the Example are not Pareto optimal as they are Pareto dominated by the unique Walras equilibrium. Ginsburgh (1994) described this last feature of the Cournot-Walras model observing that, in this model, "[...]on their "own behalf," firms (indirectly) set prices, take some surplus away from consumers who own them and prevent the economy from achieving a Pareto optimal equilibrium. Why would consumers, be stupid enough to fool themselves?" (see p. 223).

This observation is related to the problem of the rationality, in terms of consumers' preferences, of the maximization of monetary profits as a decision criterion for the firms. It is well known that, under perfect competition, the consumers unanimously agree on the maximization of monetary profits of the firms they own as shareholders, which is, therefore, their only rational decision criterion (see, for instance, DeAngelo (1981)). The problem of the rationality of the maximization of monetary profits as a decision for the firms in the model of Gabszewicz and Vial (1972) was raised by a referee of their original article which they reported as follows: "Consider a firm owned by many consumers, all of whom are identical. Given the strategies of the other firms in the economy, this firm chooses an output vector so as to maximize the wealth of each of its consumers. However, it is possible that this firm could choose a different strategy which would result in slightly lower wealth, but in a much lower price of some particular commodity which is greatly desired by the owners of the firm. Thus this alternative strategy might yield greater real income to the firms owners" (see p. 395).

We use now the same structure of a production economy as that considered in the Example of Section 3 to provide a proof that the maximization of monetary profit may not be a well-founded rationality criterion for the firms in Gabszewicz and Vial's model.

Example [Continued]**.** *Consider the production economy specified above. Moreover, consider the normalized price function p^β and the feasible production plan of firm 2, $y_2^* = (\frac{110}{17}, \frac{30}{17})$. Then, the maximization of the profit function $p^\beta(y_1, y_2^*)y_1$ is not a rational decision criterion for firm 1.*

Proof. From the previous results, we have that $y_1^* = (\frac{30}{17}, \frac{110}{17})$ is the unique feasible production plan which maximizes the profit function of firm 1, $p^\beta(y_1, y_2^*)y_1$, on G_1 and that the pair $(p^{\beta *}, (x_1^*, x_2^*))$ where $p^{\beta *} = (p^\beta(y_1^*, y_2^*)) = (\frac{1}{2}, \frac{1}{2})$ and $(x_1^*, x_2^*) = ((\frac{35}{17}, \frac{105}{17}), (\frac{105}{17}, \frac{35}{17}))$ is a Walras equilibrium relative to (y_1^*, y_2^*). Consider the feasible production plan of firm 1, $\bar{y}_2 = (1, 8)$. The pair $(\bar{p}^\beta, (\bar{x}_1, \bar{x}_2))$ where $\bar{p}^\beta = (p^\beta(\bar{y}_1, y_2^*)) = (\frac{226}{387}, \frac{161}{387})$ and $(\bar{x}_1, \bar{x}_2) = ((\frac{757}{452}, \frac{2271}{322}), (\frac{44535}{7684}, \frac{14845}{5474}))$ is a Walras equilibrium relative to $(\bar{y}_1, y_2^*)$ as

$$\bar{x}_{11}+\bar{x}_{12} = x_{11}(\bar{p}^\beta, \bar{p}^\beta\bar{y}_1)+x_{12}(\bar{p}^\beta, \bar{p}^\beta y_2^*) = \frac{757}{452}+\frac{44535}{7684} = 1+\frac{110}{17} = \bar{y}_{11}+y_{12}^*$$

and

$$\bar{x}_{21}+\bar{x}_{22} = x_{21}(\bar{p}^\beta, \bar{p}^\beta\bar{y}_1)+x_{22}(\bar{p}^\beta, \bar{p}^\beta y_2^*) = \frac{2271}{322}+\frac{14845}{5474} = 8+\frac{30}{17} = \bar{y}_{21}+y_{22}^*.$$

Then, the maximization of the profit function $p^\beta(y_1, y_2^*)y_1$ is not a rational decision criterion for firm 1 as

$$u_1(\bar{x}_1) = \frac{757(\frac{3}{161})^{\frac{3}{4}}}{2(226)^{\frac{1}{4}}} > \frac{35(3)^{\frac{3}{4}}}{17} = u_1(x_1^*).$$

■

The result is consistent with the case described by the referee quoted by Gabszewicz and Vial (1972). Indeed, consumer 1 can be considered as representative of a continuum of identical consumer in the interval $[0, 1]$, endowed with the Lebesgue measure. Given the feasible production plan of firm 2, y_2^*, the wealth of these consumers is lower at the feasible production plan $\bar{y}_1$ than at the feasible production plan y_1^* which is the unique maximum point of the profit function of firm 1. However, the price of good 2, which is "greatly desired" by the owners of firm 1, is also lower at the feasible production plan $\bar{y}_1$ than at the feasible production plan y_1^* as

$$\bar{p}^\beta = \frac{161}{387} < \frac{1}{2} = p^{\beta *}.$$

Thus, as anticipated by the referee, the feasible production plan $\bar{y}_1$ yields greater "real income" to the owners of the firm measured in terms of a greater utility level.

From the counterexample to their analysis raised by the quoted referee's report, Gabszewicz and Vial (1972) drew the conclusion that their "[...] analysis may not apply if firms are "owned" by "similar" consumers who have agreed beforehand on some unanimous preference ordering" (see p. 396). However, some years later Dierker and Grodal (1986) argued that, when each firm is owned by exactly one consumer, the analysis of Gabszewicz and Vial (1972) can be "amended" by replacing the maximization of the indirect utility of each consumer-owner instead of profit maximization as a decision criterion for the firms. They sketched a model of this particular configuration of the analysis proposed by Gabszewicz and Vial (1972) and of their alternative behavioral assumption which was generalized by Grodal (1996) who explicitly introduced the notion of Utility-Cournot-Walras equilibrium as the appropriate equilibrium concept in this framework. We now define the notion of Utility-Cournot-Walras equilibrium in the particular configuration of the Gabszewicz and Vial model considered by Dierker and Grodal (1986) and Grodal (1996).

Consider the production economy introduced in Section 2.

We assume that $n = m$, i.e., that the number of consumers is equal to the number of firms and that, for each consumer $i = 1, \dots, n$, $\theta_{ij} = 1$, if $i = j$, and $\theta_{ij} = 0$, if $i \neq j$, for each firm $j = 1, \dots, n$, i.e., each firm is owned by only one consumer.

There exists a continuous utility function u_i which represents the preference relation $\succsim_i$ as $\succsim_i$ is rational and continuous, for each consumer i.

Given feasible production plans $(y_1, \dots, y_n)$, the demand function $x_i(p, p(\omega_i + y_i))$ is well defined as $\succsim_i$ is rational, continuous, strongly monotone, and strictly convex, for each consumer i.

Given a price selection p, the indirect utility function of the owner of firm i is the function $v_i(p(y_1, \dots, y_n), p(y_1, \dots, y_n)(\omega_i + y_i)) = u_i(x_i(p(y_1, \dots, y_n), p(y_1, \dots, y_n)(\omega_i + y_i)))$, for all feasible production plans $(y_1, \dots, y_n)$.

Since the indirect utility of the owner of firm i is homogeneous of degree zero in prices, it only depends on the price selection p but not on the normalization rule.

Given a price selection p, a n-tuple of feasible production plans $(\check{y}_1, \dots, \check{y}_n)$ is a Utility-Cournot equilibrium for p if

$$\begin{aligned} &v_i(p(\check{y}_1, \dots, \check{y}_j \dots, \check{y}_n), p(\check{y}_1, \dots, \check{y}_j \dots, \check{y}_n)\check{y}_j) \\ &\geq v_i(p(\check{y}_1, \dots, y_j \dots, \check{y}_n), p(\check{y}_1, \dots, y_j \dots, \check{y}_n)y_j), \end{aligned}$$

for each $y_j \in G_j$ and for each consumer $i = 1, \dots, n$.

A Utility-Cournot-Walras equilibrium is a triplet $(p, (\check{x}_1, \dots, \check{x}_n), (\check{y}_1, \dots, \check{y}_n))$ consisting of a price selection p, a n-tuple of feasible production plans $(\check{y}_1, \dots, \check{y}_n)$, and an equilibrium allocation $(\check{x}_1, \dots, \check{x}_n)$ relative to $(\check{y}_1 \dots, \check{y}_n)$

such that the pair $(p(\check{y}_1,\ldots,\check{y}_n),(\check{x}_1,\ldots,\check{x}_n))$ is a Walras equilibrium relative to $(\check{y}_1,\ldots,\check{y}_n)$ and $(\check{y}_1,\ldots,\check{y}_n)$ is a Utility-Cournot equilibrium for p.

We can now compute a Utility-Cournot-Walras equilibrium in the same basic framework of a production economy considered in the Example in Section 3.

Example [Continued]. *Consider the production economy specified above. Then, the triplet* $(p,(\check{x}_1,\check{x}_2),(\check{y}_1,\check{y}_2))$, *where* $p(y_1,y_2) = (\frac{y_{21}+3y_{22}}{3y_{11}+y_{12}+y_{21}+3y_{22}},\frac{3y_{11}+y_{12}}{3y_{11}+y_{12}+y_{21}+3y_{22}})$, *for all feasible production plans* (y_1,y_2), $(\check{x}_1,\check{x}_2) = ((\frac{9}{4},\frac{27}{4}),(\frac{27}{4},\frac{9}{4}))$, $(\check{y}_1,\check{y}_2) = ((1,8),(8,1))$, *is a Utility-Cournot-Walras equilibrium.*

Proof. Consider the triplet $(p,(\check{x}_1,\check{x}_2),(\check{y}_1,\check{y}_2))$, where $p(y_1,y_2) = (\frac{y_{21}+3y_{22}}{3y_{11}+y_{12}+y_{21}+3y_{22}},\frac{3y_{11}+y_{12}}{3y_{11}+y_{12}+y_{21}+3y_{22}})$, for all feasible production plans (y_1,y_2), $(\check{x}_1,\check{x}_2) = ((\frac{9}{4},\frac{27}{4}),(\frac{27}{4},\frac{9}{4}))$, $(\check{y}_1,\check{y}_2) = ((1,8),(8,1))$. p is the unique price selection by the previous argument. Let $\check{p} = p(\check{y}_1,\check{y}_2) = (\frac{1}{2},\frac{1}{2})$. The pair $(\check{p},(\check{x}_1,\check{x}_2))$ is a Walras equilibrium relative to $(\check{y}_1,\check{y}_2)$ as

$$\check{x}_{11}+\check{x}_{12} = x_{11}(\check{p},\check{p}\check{y}_1)+x_{12}(\check{p},\check{p}\check{y}_2) = \frac{9}{4}+\frac{27}{4} = 1+8 = \check{y}_{11}+\check{y}_{12}$$

and

$$\check{x}_{21}+\check{x}_{22} = x_{21}(\check{p},\check{p}\check{y}_1)+x_{22}(\check{p},\check{p}\check{y}_2) = \frac{27}{4}+\frac{9}{4} = 8+1 = \check{y}_{21}+\check{y}_{22}.$$

The indirect utility function of consumer 1, given the feasible production plan on firm 2, $\check{y}_2$, is

$$v_1(p(y_1,\check{y}_2),p(y_1,\check{y}_2)y_1)) = \frac{4y_{11}y_{21}+3y_{11}+8y_{21}}{4}\left(\frac{1}{y_{21}+3}\right)^{\frac{1}{4}}\left(\frac{3}{3y_{11}+8}\right)^{\frac{3}{4}}.$$

Let $\bar{v}_1$ denote the extension of this indirect utility function to R^2_+. $\bar{v}_1$ is strictly increasing in y_{11} and y_{21} as it is straightforward to verify that $\frac{\partial\bar{v}_1(y_1,\check{y}_2)}{\partial y_{11}} > 0$ and $\frac{\partial\bar{v}_1(y_1,\check{y}_2)}{\partial y_{21}} > 0$, for each $y_1 \in R^2_+$. Moreover, it is also possible to verify, through some more cumbersome computations, that

$$-\left(\frac{\partial\bar{v}_1(y_1,\check{y}_2)}{\partial y_{11}}\right)^2\frac{\partial^2\bar{v}_1(y_1,\check{y}_2)}{\partial y_{21}^2}-\left(\frac{\partial\bar{v}_1(y_1,\check{y}_2)}{\partial y_{21}}\right)^2\frac{\partial^2\bar{v}_1(y_1,\check{y}_2)}{\partial y_{11}^2}$$
$$+2\frac{\partial\bar{v}_1(y_1,\check{y}_2)}{\partial y_{11}}\frac{\partial\bar{v}_1(y_1,\check{y}_2)}{\partial y_{21}}\frac{\partial^2\bar{v}_1(y_1,\check{y}_2)}{\partial y_{11}\partial y_{12}} > 0,$$

for each $y_1 \in R^2_+$. Then, the function $\bar{v}_1$ is strictly quasi-concave on G_1. At $(\check{y}_{11}, \check{y}_{21}) = (1, 8)$, $\check{\lambda}_1 = 0$, $\check{\lambda}_2 = \frac{19}{352}(3)^{\frac{3}{4}}$, and $\check{\lambda}_3 = \frac{59}{352}(3)^{\frac{3}{4}}$, the Kuhn-Tucker conditions for the maximization of the function $\bar{v}_1$ on G_1, which reduce to

$$y_{11}(\frac{\partial \bar{v}_1(y_1, \check{y}_2)}{\partial y_{11}} - \lambda_1 - 2\lambda_3) = 0,$$

$$y_{21}(\frac{\partial \bar{v}_1(y_1, \check{y}_2)}{\partial y_{21}} - \lambda_2 - \lambda_3) = 0,$$

$$\lambda_1(y_{11} - 2) = 0,$$

$$\lambda_2(y_{21} - 8) = 0,$$

$$\lambda_3(2y_{11} + y_{21} - 10) = 0,$$

are satisfied as $\frac{\partial \bar{v}_1(\check{y}_1, \check{y}_2)}{\partial y_{11}} = \frac{59}{176}(3)^{\frac{3}{4}}$ and $\frac{\partial \bar{v}_1(\check{y}_1, \check{y}_2)}{\partial y_{21}} = \frac{39}{176}(3)^{\frac{3}{4}}$. Then, $(\check{y}_{11}, \check{y}_{21})$ is the unique feasible production plan which maximizes $\bar{v}_1$ on G_1 as $\bar{v}_1$ is strictly quasi-concave. The indirect utility function of consumer 2, given the feasible production plan on firm 1, $\check{y}_1$, is

$$v_2(p(\check{y}_1, y_2), p(\check{y}_1, y_2)y_2)) = \frac{4y_{12}y_{22} + 8y_{12} + 3y_{22}}{4} \left(\frac{1}{y_{12} + 3}\right)^{\frac{1}{4}} \left(\frac{3}{3y_{22} + 8}\right)^{\frac{3}{4}}.$$

Let $\bar{v}_2$ denote the extension of this indirect utility function to R^2_+. Then, by using, *mutatis mutandis*, the previous argument, it is straightforward to verify that $(\check{y}_{12}, \check{y}_{22})$ is the unique feasible production plan which maximizes $\bar{v}_2$ on G_2. Hence, the triplet $(p, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$ is a Utility-Cournot-Walras equilibrium. ■

The Utility-Cournot-Walras equilibrium $(p, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$ coincides with the unique Walras equilibrium $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$ as $\hat{p} = p(\check{y}_1, \check{y}_2)$, $(\hat{x}_1, \hat{x}_2) = (\check{x}_1, \check{x}_2)$, and $(\hat{y}_1, \hat{y}_2) = (\check{y}_1, \check{y}_2)$ and, then, it is Pareto optimal.

Gabsszewicz and Vial (1972) argued, without providing a proof, that the unique Walras equilibrium of their main example "[...] is not a Cournot-Walras equilibrium" (see p. 387). However, adapting the proof of Proposition 1 in Grodal (1996) to our version of their example, we shall show that there exists a normalization rule *à la* Grodal that determines a Cournot-Walras equilibrium which coincides with the Utility-Cournot-Walras equilibrium $(p, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$ and hence with the unique Walras equilibrium $(\hat{p}, (\hat{x}_1, \hat{x}_2), (\hat{y}_1, \hat{y}_2))$.

According to Grodal (1996), given a continuous price selection p, the feasible production plans $(\bar{y}_1, \ldots, \bar{y}_m)$ are said to be p-dominated if, for a firm j, there exists a feasible production plan $y'_j \in Y_j$ such that $p(\bar{y}_1, \ldots, y'_j, \ldots, \bar{y}_m) = p(\bar{y}_1, \ldots, \bar{y}_j, \ldots, \bar{y}_m)$ and $p(\bar{y}_1, \ldots, y'_j, \ldots, \bar{y}_m)y'_j > p(\bar{y}_1, \ldots, \bar{y}_j, \ldots, \bar{y}_m)\bar{y}_j$. It is immediate to verify that a price selection p and feasible production plans $(\bar{y}_1, \ldots, \bar{y}_m)$ which are p-dominated cannot belong to a triplet which is a Cournot-Walras equilibrium or a Utility-Cournot-Walras equilibrium.

We now apply to our basic framework the argument of Proposition 1 in Grodal (1996).

Example [Continued]**.** *Consider the production economy specified above. Moreover, consider the Utility-Cournot-Walras equilibrium* $(p, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$. *Then, there exists a normalization rule à la Grodal* θ *such that the triplet* $(p^\theta, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$ *is a Cournot-Walras equilibrium.*

Proof. Consider the Utility-Cournot-Walras equilibrium $(p, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$. Clearly, the feasible production plans $(\check{y}_1, \check{y}_2)$ are not p-dominated. Moreover, we have that, at the Utility-Cournot-Walras equilibrium, the profits of both firms are strictly positive as $\check{p}\check{y}_1 = \frac{9}{2} = \check{p}\check{y}_2 > 0$. Therefore, the assumptions of Proposition 1 in Grodal (1996) are satisfied and we can apply her argument in order to build a normalization rule θ. Consider firm 1. Let $Q_1 = \{q \in [0,1] : p_1(y_1, \check{y}_2) = q$, for some $y_1 \in G_1\}$. Then, it is straightforward to show that $Q_1 = [\frac{3}{17}, \frac{11}{19}]$. Let ν_1 be a function defined on $[0,1]$ with values in R_+ such that $\nu_1(q) = -9q + 9$, for each $q \in [0, \frac{1}{2})$, and $\nu_1(q) = 9q$, for each $q \in [\frac{1}{2}, 1]$. Then, $\nu_1(q)$ is continuous. Moreover, we have that $\nu_1(q) \geq \sup\{qy_{11} + (1-q)y_{21} : p_1(y_1, \check{y}_2) = q\}$, for each $q \in Q_1$, as $\nu_1(q) = -9q + 9 \geq 19q - 3 = \sup\{qy_{11} + (1-q)y_{21} : p_1(y_1, \check{y}_2) = q\}$, for each $q \in [\frac{3}{17}, \frac{9}{23})$, $\nu_1(q) = -9q + 9 \geq \frac{-73q^2+71q-6}{q+2} = \sup\{qy_{11} + (1-q)y_{21} : p_1(y_1, \check{y}_2) = q\}$, for each $q \in [\frac{9}{23}, \frac{1}{2})$, $\nu_1(q) = 9q \geq \frac{-43q+35}{3} = \sup\{qy_{11} + (1-q)y_{21} : p_1(y_1, \check{y}_2) = q\}$, for each $q \in [\frac{1}{2}, \frac{11}{19}]$, $\nu_1(\check{p}_1) = \frac{9}{2} = \check{p}\check{y}_1$, and $\nu_1(q) \geq \frac{9}{2} = \nu_1(\check{p}_1)$, for each $q \in [0,1]$. Consider firm 2. Let $Q_2 = \{q \in [0,1] : p_1(\check{y}_1, y_2) = q$, for some $y_2 \in G_2\}$. Then, it is straightforward to show that $Q_2 = [\frac{8}{19}, \frac{14}{17}]$. Let ν_2 be a function defined on $[0,1]$ with values in R_+ such that $\nu_2(q) = -9q + 9$, for each $q \in [0, \frac{1}{2})$, and $\nu_2(q) = 9q$, for each $q \in [\frac{1}{2}, 1]$. Then, $\nu_2(q)$ is continuous. Moreover, we have that $\nu_2(q) \geq \sup\{qy_{12} + (1-q)y_{22} : p_1(\check{y}_1, y_2) = q\}$, for each $q \in Q_2$, as $\nu_2(q) = -9q+9 \geq \frac{43q-8}{3} = \sup\{qy_{11}+(1-q)y_{21} : p_1(\check{y}_1, y_2) = q\}$, for each $q \in [\frac{8}{19}, \frac{1}{2})$, $\nu_2(q) = 9q \geq \frac{-73q^2+75q-8}{3-q} = \sup\{qy_{11} + (1-q)y_{21} : p_1(\check{y}_1, y_2) = q\}$, for each $q \in [\frac{1}{2}, \frac{14}{23})$, $\nu_2(q) = 9q \geq -19q + 16 = \sup\{qy_{11} + (1-q)y_{21} : p_1(\check{y}_1, y_2) = q\}$ for each $q \in [\frac{14}{23}, \frac{14}{17}]$, $\nu_2(\check{p}_1) = \frac{9}{2} = \check{p}\check{y}_2$, and $\nu_2(q) \geq \frac{9}{2} = \nu_2(\check{p}_1)$, for each $q \in [0,1]$. Let $\rho(q)$ be a function defined on $[0,1]$ with values in

R_+ such that $\rho(q) = q(1-q)$, for each $q \in [0,1]$. Then, it is immediate to verify that ρ has a unique maximum in $q = \frac{1}{2} = \check{p}_1$. Consider the rule $\theta(y_1, y_2, p) = \frac{\rho(p_1)}{v_1(p_1)v_2(p_1)}p$, for all feasible production plans (y_1, y_2) and for each $p \in \Delta$. θ is a normalization rule *à la* Gabszewicz and Vial as $\theta(y_1, y_2, p) = \sum_{h=1}^{2} \theta_h(y_1, y_2, p)p$, for all feasible production plans (y_1, y_2) and for each $p \in \Delta$. Moreover, it is a normalization rule *à la* Grodal as $\theta(y_1, y_2, p) = \theta(y_1', y_2', p)$, for all feasible production plans (y_1, y_2) and (y_1', y_2') and for each $p \in \Delta$. Consider the normalized price function p^θ such that $p^\theta(y_1, y_2) = \theta(y_1, y_2, p(y_1, y_2)) = \frac{\rho(p_1)}{\nu_1(p_1)\nu_2(p_1)}p(y_1, y_2)$, for all feasible production plans (y_1, y_2). We have that

$$p^\theta(\check{y}_1, \check{y}_2)\check{y}_1 = \frac{\rho(\check{p}_1)}{\nu_1(\check{p}_1)\nu_2(\check{p}_1)}\check{p}\check{y}_1 = \frac{\rho(\check{p}_1)}{\nu_2(\check{p}_1)} \geq \frac{\rho(p_1(y_1, \check{y}_2))}{\nu_2(p_1(y_1, \check{y}_2))}$$
$$\geq \frac{\rho(p_1(y_1, \check{y}_2))}{\nu_1(p_1(y_1, \check{y}_2))\nu_2(p_1(y_1, \check{y}_2))}p(y_1, \check{y}_2)y_1 = p^\theta(y_1, \check{y}_2)y_1,$$

for each $y_1 \in G_1$, and

$$p^\theta(\check{y}_1, \check{y}_2)\check{y}_2 = \frac{\rho(\check{p}_1)}{\nu_1(\check{p}_1)\nu_2(\check{p}_1)}\check{p}\check{y}_2 = \frac{\rho(\check{p}_1)}{\nu_2(\check{p}_1)} \geq \frac{\rho(p_1(\check{y}_1, y_2))}{\nu_2(p_1(\check{y}_1, y_2))}$$
$$\geq \frac{\rho(p_1(\check{y}_1, y_2))}{\nu_1(p_1(\check{y}_1, y_2))\nu_2(p_1(\check{y}_1, y_2))}p(\check{y}_1, y_2)y_2 = p^\theta(\check{y}_1, y_2)y_2,$$

for each $y_2 \in G_2$, as $\nu_j(\check{p}_1) = \check{p}\check{y}_j$, ρ has a unique maximum in $\check{p}_1$, $\nu_j(q) \geq \nu_j(\check{p}_1)$, for each $q \in [0,1]$, and $\nu_j(q) \geq \sup\{qy_{11} + (1-q)y_{21} : p_1(y_1, \check{y}_2) = q\}$, for each $q \in Q_j$, for each firm $j = 1, 2$. Therefore, the pair of feasible production plans $(\check{y}_1, \check{y}_2)$ is a Cournot equilibrium for p^θ. Hence, the triplet $(p^\theta, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$ is a Cournot-Walras equilibrium. ∎

The Cournot-Walras equilibrium $(p^\theta, (\check{x}_1, \check{x}_2), (\check{y}_1, \check{y}_2))$ coincides with the unique Walras equilibrium as $p^\theta(\check{y}_1, \check{y}_2) = (\frac{1}{9}, \frac{1}{9}) = \frac{2}{9}\hat{p}$, $(\hat{x}_1, \hat{x}_2) = (\check{x}_1, \check{x}_2)$, and $(\hat{y}_1, \hat{y}_2) = (\check{y}_1, \check{y}_2)$ and, then, it is Pareto optimal.

5 Conclusion

In this paper, we have reviewed the main theoretical issues related to the concept of Cournot-Walras equilibrium introduced by Gabszewicz and Vial (1972) using, as a starting point, their own main example. This review has led to a surprising result due to the indeterminacy generated by normalization rules: In the Gabszewicz and Vial model, Cournotian duopolistic firms may be Walrasian.

Recently, Azar and Vives (2021) noticed that "Oligopoly is widespread and allegedly on the rise. Many industries are characterized by oligopolistic conditions, including, but not limited to, the digital ones dominated by GAFAM: Google (now Alphabet), Apple, Facebook, Amazon, and Microsoft. These firms, as well as others, have influence in the aggregate economy" (see p. 1). This observation led these authors to reconsider the general equilibrium analysis *à la* Cournot introduced by Gabszewicz and Vial (1972) in order to appropriately capture some features of oligopolistic interaction in a model of interrelated markets with a macroeconomic flavor. Their paper, which was motivated by a huge empirical evidence showing an upsurge in oligopoly in the real word economy, might be considered as the initial piece of a parallel upsurge in theoretical general equilibrium models of oligopoly. Azar and Vives (2021) considered particular production economies in which each firm is owned by many heterogeneous shareholders.

Gabszewicz and Vial (1972), after having considered the criticism to profit maximization as a rational criterion for the firms, already observed that their analysis "may not apply if some firms are "owned" by "similar" consumers who have agreed beforehand on some unanimous preference ordering. By contrast, if all firms are owned by many "different" consumers, the impossibility of aggregating their various preferences justifies, by default and as a first approximation, the use of monetary profits as an objective for these firms" (see p. 396). We have seen that, in the case where each firm is owned by a consumer, or a continuum of identical consumers, both the related problems of price normalization and the rationality of the profit criterion, can be overcome using the notion of Utility-Cournot-Walras equilibrium proposed by Grodal (1996). We have also shown that, due to the indeterminacy result proved by Grodal (1996), there exists a normalization rule such that a Utility-Cournot-Walras equilibrium is also a Cournot-Walras equilibrium at which profit maximization is a rational criterion. This reverses the claim against their own theory formulated by Gabszewicz and Vial (1972) which we have quoted above. In the same quotation, Gabszewicz and Vial (1972) considered the criterion of profit maximization acceptable as a "first approximation" when each firm is owned by many different shareholders, whereas Grodal (1996) argued that, in this case, given the indeterminacy generated by normalization rules, "[...] there is no natural objective function for the firm" (p. 19).

Azar and Vives (2021) proposed to overcome this theoretical deadlock assuming that, in their specific model, "firm j's objective function is to maximize a weighted average of the (indirect) utilities of its owners, where the weights are proportional to the numbers of shares. In other words, we suppose that ownership confers control in proportion to the shares owned" (p. 1008). Nevertheless, at this stage, Grodal (1996) would object that "If a firm has an objective function which is related to the preferences of its shareholders one might also obtain that markets in shares of firms will be active in equilibrium" (p. 21)...! Demichelis and Ritzberger (2011) followed this suggestion and proposed an approach "[...] to include an analysis of the institutions that regulate investors' control over firms. This, of course, transcends general equilibrium theory, that is meant to be "institution-free," as it requires an explicit model of what determines corporate control" (p. 222). The story continued through other papers and could continue in the future.

Given the growing importance of oligopolistic interaction in interrelated markets reminded above and the theoretical issues reviewed in this paper, we think that further research should move in the shadow line between partial and general equilibrium theory as we believe that some theory is better that no theory at all.

References

[1] Azar J. Vives X. (2021), "General equilibrium oligopoly and ownership structure," *Econometrica* **89**, 999-1048.

[2] Böhm V. (1994), "The foundation of the theory of monopolistic competition revisited," *Journal of Economic Theory* **63**, 208-218.

[3] Bonanno G. (1990), "General equilibrium theory with imperfect competition," *Journal of Economic Surveys* **4**, 297-328

[4] Busetto F., Codognato G., Julien L. (2020), "Atomic Leontievian Cournotian traders are always Walrasian," *Games and Economic Behavior* **122**, 318-327.

[5] Codognato G., Gabszewicz J.J. (1991), "Équilibres de Cournot-Walras dans une économie d'échange. *Revue Économique* **42**, 1013-1026.

[6] Codognato G., Ghosal S., Tonin S. (2015), "Atomic Cournotian traders may be Walrasian,", *Journal of Economic Theory* **159**, 1-14.

[7] DeAngelo H. (1981), "Competition and unanimity," *American Economic Review* **71**, 18-27.

[8] Demichelis S., Ritzberger K. (2011), "A general equilibrium analysis of corporate control and the stock market," *Economic Theory* **46**, 22

[9] Dierker H., Grodal B. (1986), "Nonexistence of Cournot-Walras equilibrium in a general equilibrium model with two oligopolists," in Hildenbrand W., Mas-Colell A. (eds), *Contributions to mathematical economics in honor of Gérard Debreu*, North-Holland, Amsterdam.

[10] Gabszewicz J.J., Vial J.P. (1972), "Oligopoly "à la Cournot" in a general equilibrium analysis," *Journal of Economic Theory* **4**, 381-400.

[11] Ginsburgh V. (1994), "In the Cournot-Walras general equilibrium model, there may be "more to gain" by changing the numéraire than by eliminating imperfections: a two-good economy example," in Mercenier J., Srinivisan T.N. (eds), *Applied general equilibrium and economic development*, The University of Michigan Press, Ann Arbor.

[12] Grodal B. (1996), "Profit maximization and imperfect competition," in Allen B. (ed), *Economics in a changing world*, Macmillan, London.

[13] Hart O. (1985). "Imperfect competition in general equilibrium: an overview of recent work," in Arrow K.J., Honkapohja S. (eds), *Frontiers of economics*, Blackwell, Oxford.

[14] Mas-Colell A. (1982), "The Cournotian foundations of Walrasian equilibrium theory," in Hildenbrand W. (ed), *Advances in economic theory*, Cambridge University Press, Cambridge.