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Abstract

We consider a bilateral oligopoly version of the Shapley window model with large traders, represented as atoms, and small traders, represented by an atomless part. For this model, we show that, when atoms have Leontievian utility functions, any Walras allocation is a Cournot-Nash allocation. This result, together with the main theorem proved in Busetto et al. (2020), implies the equivalence between the set of Cournot-Nash allocations and the set of Walras allocations.
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1 Introduction

Busetto et al. (2020) considered the mixed version of the bilateral oligopoly model introduced by Codognato et al. (2015): a mixed exchange economy à la Shitovitz (1973) where large traders are represented as atoms and small traders are represented by an atomless part; noncooperative exchange is formalized as in the Shapley window model, a strategic market game with complete markets which was first proposed informally by Lloyd S. Shapley

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and further studied by Sahi and Yao (1989), Codognato and Ghosal (2000), and Busetto et al. (2011), among others.

In this framework, they proved a theorem which implies that, when traders in the atomless part have continuous, strongly monotone, and quasi-concave utility functions whereas atoms have Leontievian utility functions, any Cournot-Nash allocation is a Walras allocation.

In this note, we prove a theorem which implies that, in the bilateral oligopoly model considered by Busetto et al. (2020), any Walras allocation is a Cournot-Nash allocation. Moreover, we straightforwardly show that this theorem and that proved by Busetto et al (2020) imply the equivalence between the set of Cournot-Nash allocations and the set of Walras allocations.

In Section 2, we introduce the mathematical model. In section 3, we prove our main theorem. In Section 4, we conclude.

2 The mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and T_0 the atomless part of T. We assume that $\mu(T_1) > 0$ and $\mu(T_0) > 0$. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. A coalition is a nonnull element of $\mathcal{T}$. The word "integrable" is to be understood in the sense of Lebesgue.

There are two different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. *There is a coalition S such that $\mathbf{w}^1(t) > 0$, $\mathbf{w}^2(t) = 0$, for each $t \in S$, $\mathbf{w}^1(t) = 0$, $\mathbf{w}^2(t) > 0$, for each $t \in S^c$. Moreover, card$(S \cap T_1) \geq 2$, whenever $\mu(S \cap T_0) = 0$, and card$(S^c \cap T_1) \geq 2$, whenever $\mu(S^c \cap T_0) = 0$.*[1]

[1] card(A) denotes the cardinality of a set A.

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^2_+ \to R$ is continuous, strongly monotone, and quasi-concave, for each $t \in T_0$, and $u_t(x^1, x^2) = \min\{a_{t1}x^1, a_{t2}x^2\}$, with $a_{t1} > 0$ and $a_{t2} > 0$, for each $t \in T_1$.*

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

A price vector is a nonnull vector $p \in R^2_+$. A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector $p^* \gg 0$ and an allocation $\mathbf{x}^*$ such that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$ and $u_t(\mathbf{x}^*(t)) \geq u_t(y)$, for all $y \in \{x \in R^2_+ : p^*x = p^*\mathbf{w}(t)\}$, for each $t \in T$. A Walras allocation is an allocation $\mathbf{x}^*$ for which there exists a price vector p^* such that the pair $(p^*, \mathbf{x}^*)$ is a Walras equilibrium.

Borrowing from Codognato et al. (2015), we introduce now the two-commodity version of the Shapley window model. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^4_+)$ such that, for each $t \in T$, $\mathbf{B}(t) = \{(b_{ij}) \in R^4_+ : \sum_{j=1}^2 b_{ij} \leq \mathbf{w}^i(t),\ i = 1, 2\}$. With some abuse of notation, we denote by $b(t) \in \mathbf{B}(t)$ a strategy of trader t, where $b_{ij}(t)$, $i, j = 1, 2$, represents the amount of commodity i that trader t offers in exchange for commodity j. A strategy selection is an integrable function $\mathbf{b} : T \to R^4_+$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. Given a strategy selection $\mathbf{b}$, we call aggregate matrix the matrix $\bar{\mathbf{B}}$ such that $\bar{\mathbf{b}}_{ij} = (\int_T \mathbf{b}_{ij}(t)\, d\mu)$, $i, j = 1, 2$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ the strategy selection obtained from $\mathbf{b}$ by replacing $\mathbf{b}(t)$ with $b(t) \in \mathbf{B}(t)$ and by $\bar{\mathbf{B}} \setminus b(t)$ the corresponding aggregate matrix.

Consider the following two further definitions (see Sahi and Yao (1989)).

Definition 1. *A nonnegative square matrix D is said to be irreducible if, for every pair (i, j), with $i \neq j$, there is a positive integer k such that $d^{(k)}_{ij} > 0$, where $d^{(k)}_{ij}$ denotes the ij-th entry of the k-th power D^k of D.*

Definition 2. *Given a strategy selection $\mathbf{b}$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \sum_{i=1}^2 p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^2 \bar{\mathbf{b}}_{ji}),\ j = 1, 2. \tag{1}$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p(\mathbf{b})$ a function which associates with each strategy selection $\mathbf{b}$ the unique, up to a scalar multiple, price vector p satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and is equal to 0 otherwise.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{b}_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}_{ij}(t)\frac{p^i}{p^j}, \text{ if } p \in R^2_{++},\\ \mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,}\end{aligned}$$

$j = 1, 2$, for each $t \in T$.

Given a strategy selection $\mathbf{b}$ and the function $p(\mathbf{b})$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),$$

for each $t \in T$. It is straightforward to show that this assignment is an allocation satisfying the budget constraint $p(\mathbf{b})\mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})) = p(\mathbf{b})\mathbf{w}(t)$, for each $t \in T$.

We are now able to define the notion of a Cournot-Nash equilibrium for this reformulation of the Shapley window model (see Codognato and Ghosal (2000) and Busetto et al. (2011)).

Definition 3. *A strategy selection $\hat{\mathbf{b}}$ such that $\hat{\bar{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, b(t), p(\hat{\mathbf{b}} \setminus b(t)))),$$

for each $b(t) \in \mathbf{B}(t)$ and for each $t \in T$.

A Cournot-Nash allocation is an allocation $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium.

3 Walras allocations are always Cournot-Nash allocations

Busetto et al. (2020) proved the following result which implies that, in the bilateral oligopoly model described in the previous section, any Cournot-Nash allocation is a Walras allocation.

Theorem 1. *Under Assumptions 1, 2, and 3, let $\hat{\mathbf{b}}$ be a Cournot-Nash equilibrium and let $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$. Then, the pair $(\hat{p}, \hat{\mathbf{x}})$ is a Walras equilibrium.*

We state and prove now a new result which implies that, in the bilateral oligopoly model described in the previous section, any Walras allocation is a Cournot-Nash allocation.

Theorem 2. *Under Assumptions 1, 2, and 3, let $(p^*, \mathbf{x}^*)$ be a Walras equilibrium. Then, there exists a Cournot-Nash equilibrium $\mathbf{b}^*$ such that $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{b}}^*(t), p(\mathbf{b}^*))$, for each $t \in T$.*

Proof. Let $(p^*, \mathbf{x}^*)$ be a Walras equilibrium. We have that $p^*\mathbf{x}^*(t) = p^*\mathbf{w}(t)$, for each $t \in T$, by Assumption 2. Then, there exist $\lambda^{*j}(t) \geq 0$, $j = 1, 2$, $\sum_{j=1}^{2} \lambda^{*j}(t) = 1$, such that

$$\mathbf{x}^{*j}(t) = \lambda^{*j}(t)\frac{p^*\mathbf{w}(t)}{p^{*j}},$$

$j = 1, 2$, for each $t \in T$, by Lemma 5 in Codognato and Ghosal (2000). Let $\boldsymbol{\lambda}^* : T \to R^2_+$ be a function such that $\boldsymbol{\lambda}^{*j}(t) = \lambda^{*j}(t)$, $j = 1, 2$, for each $t \in T$. It is straightforward to show that the function $\mathbf{w}^i(t)\boldsymbol{\lambda}^{*j}(t)$, $i, j = 1, 2$, for each $t \in T$, is integrable on T. Let $\mathbf{b}^*$ be a strategy selection such that $\mathbf{b}^*_{ij}(t) = \mathbf{w}^i(t)\boldsymbol{\lambda}^{*j}(t)$, $i, j = 1, 2$, for each $t \in T$. Suppose that $\bar{\mathbf{b}}^*_{12} = 0$. Consider the case where $S \cap T_1 \neq \emptyset$. Consider an atom $\tau \in S$. We have that

$$\mathbf{x}^{*2}(\tau) = \frac{a_{\tau 1} p^{*1} \mathbf{w}^1(\tau)}{a_{\tau 2} p^{*1} + a_{\tau 1} p^{*2}} > 0.$$

Then, we have that

$$\boldsymbol{\lambda}^{*2}(\tau) = \frac{a_{\tau 1} p^{*2}}{a_{\tau 2} p^{*1} + a_{\tau 1} p^{*2}} > 0.$$

But then, we obtain that

$$\mathbf{b}^*_{12}(\tau) = \mathbf{w}^1(\tau)\boldsymbol{\lambda}^{*2}(\tau) > 0,$$

a contradiction. Consider the case where $S \cap T_1 = \emptyset$. Then, it must be that $S \subseteq T_0$ and $S^c \cap T_1 \neq \emptyset$ as $\mu(T_1) > 0$ and $\mu(T_0) > 0$. Consider a trader $t \in S$. We have that $\boldsymbol{\lambda}^2(t) = 0$ as $\mathbf{b}^*_{12}(t) = \mathbf{w}^1(t)\boldsymbol{\lambda}^2(t) = 0$ and $\mathbf{w}^1(t) > 0$, for each $t \in S$. Then, it must be that

$$\mathbf{x}^{*2}(t) = \lambda^2(t)\frac{p^{*1}\mathbf{w}^1(t)}{p^{*2}} = 0,$$

for each $t \in S$. Consider an atom $\tau \in S^c$. We have that

$$\mathbf{x}^{*2}(\tau) = \frac{a_{\tau 1} p^{*2} \mathbf{w}^2(\tau)}{a_{\tau 2} p^{*1} + a_{\tau 1} p^{*2}} < \mathbf{w}^2(\tau).$$

Then, we have that

$$\int_S \mathbf{x}^{*2}(t)\, d\mu + \int_{S^c} \mathbf{x}^{*2}(t)\, d\mu = \int_{S^c} \mathbf{x}^{*2}(t)\, d\mu < \int_T \mathbf{w}^2(t)\, d\mu,$$

a contradiction. We can conclude that $\bar{\mathbf{b}}^*_{12} > 0$. Using, *mutatis mutandis*, the previous argument, we can also conclude that $\bar{\mathbf{b}}^*_{21} > 0$. Therefore, the matrix $\bar{\mathbf{B}}^*$ is irreducible. Consider the assignment $\mathbf{x}(t, \hat{\mathbf{b}}^*(t), p^*)$, for each $t \in T$. We have that

$$\begin{aligned}
\mathbf{x}^j(t, \hat{\mathbf{b}}^*(t), p^*) &= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{b}^*_{ji}(t) + \sum_{i=1}^{2} \mathbf{b}^*_{ij}(t) \frac{p^{*i}}{p^{*j}} \\
&= \mathbf{w}^j(t) - \sum_{i=1}^{2} \mathbf{w}^j(t) \boldsymbol{\lambda}^{*i} + \sum_{i=1}^{2} \mathbf{w}^i(t) \boldsymbol{\lambda}^{*j} \frac{p^{*i}}{p^{*j}} \\
&= \boldsymbol{\lambda}^{*j}(t) \frac{p^* \mathbf{w}(t)}{p^{*j}} = \mathbf{x}^{*j}(t),
\end{aligned}$$

$j = 1, 2$, for each $t \in T$. Then, we obtain that

$$\int_T \mathbf{x}^{*j}(t)\, d\mu = \int_T \mathbf{w}^j(t)\, d\mu - \sum_{i=1}^{2} \bar{\mathbf{b}}^*_{ji}(t) + \sum_{i=1}^{2} \bar{\mathbf{b}}^*_{ij}(t) \frac{p^{*i}}{p^{*j}} = \int_T \mathbf{w}^j(t)\, d\mu,$$

$j = 1, 2$, as $\mathbf{x}^*$ is an allocation. But then, p^* satisfies (1) as

$$\sum_{i=1}^{2} p^{*i} \bar{\mathbf{b}}^*_{ij} = p^{*j} (\sum_{i=1}^{2} \bar{\mathbf{b}}^*_{ji}),$$

$j = 1, 2$, and, consequently, $p^* = p(\mathbf{b}^*)$, as the matrix $\bar{\mathbf{B}}^*$ is irreducible. Therefore, we have that $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{b}}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. Suppose that $\mathbf{b}^*$ is not a Cournot-Nash equilibrium. Then, there is a trader τ and a strategy $b'(\tau) \in \mathbf{B}(\tau)$ such that

$$u_\tau(\mathbf{x}(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau)))) > u_\tau(\mathbf{x}(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*))).$$

Suppose that $\tau \in T_1$. Moreover, suppose, without loss of generality, that $\mathbf{w}^1(\tau) = 0$ and $\mathbf{w}^2(\tau) > 0$. We have that

$$a_{\tau 1} \mathbf{x}^1(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*) = a_{\tau 1} \mathbf{x}^{*1}(\tau) = a_{\tau 2} \mathbf{x}^{*2}(\tau) = a_{\tau 2} \mathbf{x}^2(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*)),$$

as $\mathbf{x}^*$ is a Walras allocation. Consider the case where $b'_{21}(\tau) > \mathbf{b}^*_{21}(\tau)$. It is straightforward to verify that

$$\mathbf{x}^1(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau))) = b'_{21}(\tau)\frac{\bar{\mathbf{b}}^*_{12}}{\bar{\mathbf{b}}^*_{21} - \mathbf{b}^*_{21}(\tau)\mu(\tau) + b'_{21}(\tau)\mu(\tau)}$$
$$> \mathbf{b}^*_{21}(\tau)\frac{\bar{\mathbf{b}}^*_{12}}{\bar{\mathbf{b}}^*_{21}} = \mathbf{x}^1(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*))$$

and

$$\mathbf{x}^2(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau))) = \mathbf{w}^2(\tau) - b'_{21}(\tau)$$
$$< \mathbf{w}^2(\tau) - \mathbf{b}^*_{21}(\tau) = \mathbf{x}^2(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*)).$$

Then, we have that

$$a_{\tau 1}\mathbf{x}^1(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau))) > a_{\tau 1}\mathbf{x}^1(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*))$$
$$= a_{\tau 2}\mathbf{x}^2(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*)) > a_{\tau 2}\mathbf{x}^2(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau))).$$

But then, we obtain that

$$u_\tau(\mathbf{x}(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau)))) = a_{\tau 2}\mathbf{x}^2(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau)))$$
$$< a_{\tau 2}\mathbf{x}^2(t, \hat{\mathbf{b}}^*(t), p(\mathbf{b}^*)) = u_\tau(\mathbf{x}(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*))),$$

a contradiction. Consider the case where $b'_{21}(\tau) < \mathbf{b}^*_{21}(\tau)$. Using, *mutatis mutandis*, the previous argument, we have that

$$a_{\tau 1}\mathbf{x}^1(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau))) < a_{\tau 1}\mathbf{x}^1(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*))$$
$$= a_{\tau 2}\mathbf{x}^2(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*)) < a_{\tau 2}\mathbf{x}^2(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau))).$$

Then, it follows that

$$u_\tau(\mathbf{x}(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau)))) = a_{\tau 1}\mathbf{x}^1(\tau, b'(\tau), p(\mathbf{b}^* \setminus b'(\tau)))$$
$$< a_{\tau 1}\mathbf{x}^1(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*)) = u_\tau(\mathbf{x}(\tau, \mathbf{b}^*(\tau), p(\mathbf{b}^*))),$$

a contradiction. Suppose that $\tau \in T_0$. We have that $p(\mathbf{b}^* \setminus b'(\tau)) = p(\mathbf{b}^*) = p^*$, by Lemma 1 in Codognato and Ghosal (2000). Moreover, it is straightforward to verify that $p^*\mathbf{x}(\tau, b'(\tau), p^*)) = p^*\mathbf{w}^2(\tau)$. Then, we have that $u_\tau(\mathbf{x}(\tau, b'(\tau), p^*)) > u_\tau(\mathbf{x}^*(\tau))$ and $\mathbf{x}(\tau, b'(\tau), p^*) \in \{x \in R^2_+ : p^*x = p^*\mathbf{w}^2(\tau)\}$, a contradiction. Hence, there exists a Cournot-Nash equilibrium $\mathbf{b}^*$ such that $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{b}}^*(t), p(\mathbf{b}^*))$, for each $t \in T$. ■

Theorems 1 and 2 have the following straightforward implication concerning the equivalence between the set of Cournot-Nash allocations and the set of Walras allocations.

Corollary. *Under Assumptions 1, 2, and 3, the set of Cournot-Nash allocations coincides with the set of Walras allocations.*

Proof. Let $\hat{\mathbf{x}}$ be a Cournot-Nash allocation. Then, we have that $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))$, for each $t \in T$, where $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium. Moreover, the pair $(\hat{p}, \hat{\mathbf{x}})$, where $\hat{p} = p(\hat{\mathbf{b}})$, is a Walras equilibrium, by Theorem 1. Therefore, $\hat{\mathbf{x}}$ is a Walras allocation. Let $\mathbf{x}^*$ be a Walras allocation. Then, there exists a Cournot-Nash equilibrium $\mathbf{b}^*$ such that $\mathbf{x}^*(t) = \mathbf{x}(t, \hat{\mathbf{b}}^*(t), p(\mathbf{b}^*))$, for each $t \in T$, by Theorem 2. Therefore, $\mathbf{x}^*$ is a Cournot-Nash allocation. Hence, the set of Cournot-Nash allocations coincides with the set of Walras allocations. ∎

The following example, borrowed from Busetto et al. (2020), shows that the Corollary holds non-vacuously.

Example. *Consider the following specification of the exchange economy satisfying Assumptions 1, 2, and 3.* $T_0 = [0, 1]$, $T_1 = \{2, 3\}$, T_0 *is taken with Lebesgue measure,* $\mu(2) = \mu(3) = 1$, $\mathbf{w}(t) = (4, 0)$, $u_t(x) = \sqrt{x^1} + \sqrt{x^2}$, *for each* $t \in T_0$, $\mathbf{w}(2) = \mathbf{w}(3) = (0, 4)$, $u_2(x) = u_3(x) = \min\{x^1, x^2\}$. *Then, the allocation* $\mathbf{x}^*$ *such that* $(\mathbf{x}^{*1}(t), \mathbf{x}^{*2}(t)) = (\frac{4}{3}, \frac{16}{3})$, *for each* $t \in T_0$, $(\mathbf{x}^{*1}(2), \mathbf{x}^{*2}(2)) = (\mathbf{x}^{*1}(3), \mathbf{x}^{*2}(3)) = (\frac{4}{3}, \frac{4}{3})$ *is the unique Walras allocation, which is also the unique Cournot-Nash allocation.*

Proof. See the proof of the Example in Busetto et al. (2020). ∎

Theorem 2 in Busetto et al. (2020) proved that, under Assumption 1, 2, and 3, in the mixed bilateral oligopoly model, the set of the Cournot-Nash allocations of the Shapley window model coincides with the set of the Cournot-Nash allocations of both the model of Dubey and Shubik (1978) and its generalization proposed by Amir et al. (1990). Hence, the Corollary can be straightforwardly extended to those models.

4 Conclusion

In the framework of a mixed bilateral oligopoly, we have here proved that, when atoms have Leontievian utility functions, Walras allocations are always Cournot-Nash allocations. Moreover, we have shown that this result, together with the main theorem in Busetto et al. (2020), implies that the set of Cournot-Nash allocations coincides with the set of Walras allocations.

We remind that the equivalence theorem between the core and the set of Walras allocations proved by Shitovitz (1973) rests on the assumptions that atoms' preferences are strongly monotone. Here, we have proved the equivalence between the set of Cournot-Nash allocations and the set of Walras allocations, a result which holds when atoms' preferences are monotone but not necessarily strongly monotone. Our results should stimulate a further investigation on the validity of the core equivalence theorems when the strong monotonicity assumption is relaxed.

References

[1] Amir R., Sahi S., Shubik M., Yao S. (1990), "A strategic market game with complete markets," *Journal of Economic Theory* **51**, 126-143.

[2] Busetto F., Codognato G., Ghosal S. (2011), "Noncooperative oligopoly in markets with a continuum of traders," *Games and Economic Behavior*, **72**, 38-45.

[3] Busetto F., Codognato G., Julien L. (2020), "Atomic Leontievian Cournotian traders are always Walrasian," *Games and Economic Behavior* **122**, 318-327.

[4] Codognato G., Ghosal S. (2000), "Cournot-Nash equilibria in limit exchange economies with complete markets and consistent prices," *Journal of Mathematical Economics* **34**, 39-53.

[5] Codognato G., Ghosal S., Tonin S. (2015), "Atomic Cournotian traders may be Walrasian," *Journal of Economic Theory* **159**, 1-14.

[6] Dubey P., Shubik M. (1978), "The noncooperative equilibria of a closed trading economy with market supply and bidding strategies *Journal of Economic Theory* **17**, 1-20.

[7] Sahi S., Yao S. (1989), "The noncooperative equilibria of a trading economy with complete markets and consistent prices," *Journal of Mathematical Economics* **18**, 325-346.

[8] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.