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Abstract

In this paper, we use the integer programming approach to mechanism design, first introduced by Sethuraman et al. (2003), and then systematized by Vohra (2011), to reformulate issues concerning nondictatorial Arrovian social welfare functions with and without ties. Then, we use the integer programming to prove a new result which shows that, when the number of agents is even, a necessary and sufficient condition for the simple majority rule to be an Arrovian social welfare function is that it is defined on a domain which is echoic with antagonistic preferences. This condition is based on the definitions of echoic and antagonistic preferences in Inada (1969).

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1 Introduction

Vohra (2011) based his monograph on mechanism design on integer programming. He claimed that this approach has basically three advantages: simplicity, unity, and reach, meaning, respectively, that it may simplify arguments, unify disparate results, and solve problems which are beyond the reach of other approaches.

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In this paper, we use the evoked advantages of the integer programming approach to analyze nondictatorial Arrovian Social Welfare Functions (ASWFs) and the Simple Majority Rule (SMR). Our analysis is, to some extent, complementary to that undertaken by Sethuraman et al. (2006) about anonymous monotonic ASWFs in an integer programming framework.

Sethuraman et al. (2003) developed Integer Programs (IPs) in which variables assume values only in the set $\{0, 1\}$. These IPs were inspired by the characterization of decomposable domains introduced by Kalai and Muller (1977) and they allowed Sethuraman et al. (2003) to establish a one-to-one correspondence, on domains of antisymmetric preference orderings, between the set of feasible solutions of a binary IP and the set of ASWFs without ties.

Busetto et al. (2015) generalized the approach proposed by Sethuraman et al. (2003), specifying IPs in which variables are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$, called ternary IPs, and they established a one-to-one correspondence between the set of feasible solutions of a ternary IP, which they called IP1, and the set of ASWFs with and without ties.

Here, we use IP1 to reformulate a theorem, shown by Busetto et al. (2018), which allows to prove, as a corollary, Theorem 2 in Kalai and Muller (1977) for nondictatorial ASWFs without ties. To this end, we use the notion of decomposability introduced by Busetto et al. (2015). Moreover, we restate the notion of a strictly decomposable domain, introduced by Busetto et al. (2015), and their characterization theorem, establishing, as a corollary, that a domain of antisymmetric preference orderings admits nondictatorial ASWFs with ties if and only if it is strictly decomposable.

Then, we consider a reformulation of the Simple Majority Rule (SMR) in the framework of integer programming. We first restate the integer programming version, provided by Sethuraman et al. (2003), of a theorem proved by Sen (1966), which shows that, when the number of agents is odd, a necessary and sufficient condition for the SMR to be an ASWF is that it is defined on a domain which does not contain a Condorcet triple. This theorem characterizes the SMR as a nondictatorial ASWF without ties. Therefore, we straightforwardly show that the domains which do not contain a Condercet triple are decomposable. Then, we use IP1 to state and prove a new result which shows that, when the number of agents is even, a necessary and sufficient condition for the SMR to be an ASWF is that it is defined on a domain which is echoic with antagonistic preferences, a condition based on the definitions of echoic and antagonistic preferences in Inada (1969). This theorem characterizes the SMR as a nondictatorial ASWF with ties. Therefore, we straightforwardly show that the domains which are echoic with antagonistic preferences are strictly decomposable. Finally, we show that the set of domains admitting an ASWF with ties based on the SMR is a strict subset of the set of domains admitting an ASWF without ties based on the SMR.

The paper is organized as follows. In Section 2, we introduce the notation and the basic definitions. In Section 3, we restate the possibility theorems for nondictatorial ASWFs with and without ties, using IP1. In Section 4, we use IP1 to prove a new theorem which characterizes the SMR as a nondictatorial ASWF with ties when the number of agents is even. In Section 5, we draw some conclusions.

2 Notation and definitions

Let E be any initial finite subset of the natural numbers with at least two elements and let $|E|$ be the cardinality of E, denoted by n. Elements of E are called agents.

Let $\mathcal{E}$ be the collection of all subsets of E. Given a set $S \in \mathcal{E}$, let $S^c = E \setminus S$.

Let $\mathcal{A}$ be a set such that $|\mathcal{A}| \geq 3$. Elements of $\mathcal{A}$ are called alternatives.

Let $\mathcal{A}^2$ denote the set of all ordered pairs of alternatives.

Let $\mathcal{R}$ be the set of all the complete and transitive binary relations on $\mathcal{A}$, called preference orderings.

Let Σ be the set of all antisymmetric preference orderings.

Let Ω denote a subset of Σ such that $|\Omega| \geq 2$. An element of Ω is called admissible preference ordering and is denoted by $\mathbf{p}$. We write $x\mathbf{p}y$ if x is ranked above y under $\mathbf{p}$.

A pair $(x, y) \in \mathcal{A}^2$ is called trivial if there are not $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y$ and $y\mathbf{q}x$. Let TR denote the set of trivial pairs. We adopt the convention that all pairs $(x, x) \in \mathcal{A}^2$ are trivial.

A pair $(x, y) \in \mathcal{A}^2$ is nontrivial if it is not trivial. Let NTR denote the set of nontrivial pairs.

Let Ω^n denote the n-fold Cartesian product of Ω. An element of Ω^n is called a preference profile and is denoted by $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is the antisymmetric preference ordering of agent $i \in E$.

A Social Welfare Function (SWF) on Ω is a function $f : \Omega^n \to \mathcal{R}$.

f is said to be "without ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) = \emptyset$.

f is said to be "with ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) \neq \emptyset$.

Given $\mathbf{P} \in \Omega^n$, let $P(f(\mathbf{P}))$ and $I(f(\mathbf{P}))$ be binary relations on $\mathcal{A}$. We write $xP(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ but not $yf(\mathbf{P})x$ and $xI(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ and $yf(\mathbf{P})x$.

A SWF on Ω, f, satisfies Pareto Optimality (PO) if, for all $(x, y) \in \mathcal{A}^2$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_i y$, for all $i \in E$, implies $xP(f(\mathbf{P}))y$.

A SWF on Ω, f, satisfies Independence of Irrelevant Alternatives (IIA) if, for all $(x, y) \in NTR$ and for all $\mathbf{P}, \mathbf{P'} \in \Omega^n$, $x\mathbf{p}_i y$ if and only if $x\mathbf{p}'_i y$, for all $i \in E$, implies, $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P'})y$, and, $yf(\mathbf{P})x$ if and only if $yf(\mathbf{P'})x$.

An Arrovian Social Welfare Function (ASWF) on Ω is a SWF on Ω, f, which satisfies PO and IIA.

An ASWF on Ω, f, is dictatorial if there exists $j \in E$ such that, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_j y$ implies $xP(f(\mathbf{P}))y$. f is nondictatorial if it is not dictatorial.

Given $(x, y) \in \mathcal{A}^2$ and $S \in \mathcal{E}$, let $d_S(x, y)$ denote a variable such that $d_S(x, y) \in \{0, \frac{1}{2}, 1\}$.

An Integer Program (IP) on Ω consists of a set of linear constraints, related to the preference orderings in Ω, on variables $d_S(x, y)$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, and of the further conventional constraints that $d_E(x, y) = 1$ and $d_\emptyset(y, x) = 0$, for all $(x, y) \in TR$.

Let d denote a feasible solution (henceforth, for simplicity, only "solution") to an IP on Ω. d is said to be a binary solution if variables $d_S(x, y)$ reduce to assume values in the set $\{0, 1\}$, for all $(x, y) \in NTR$, and for all $S \in \mathcal{E}$. It is said to be a "ternary" solution, otherwise.

A solution d is dictatorial if there exists $j \in E$ such that $d_S(x, y) = 1$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, with $j \in S$. d is nondictatorial if it is not dictatorial.

An ASWF on Ω, f, and a solution to an IP on the same Ω, d, are said to correspond if, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $xP(f(\mathbf{P}))y$ if and only if $d_S(x, y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x, y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$.

3 Nondictatorial Arrovian social welfare functions and integer programming

The first formulation of an IP on Ω was proposed by Sethuraman et al. (2003), for the case where $d_S(x, y) \in \{0, 1\}$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$. Busetto et al. (2015) extended their approach, specifying an IP on Ω, called IP1, in which variables $d_S(x, y)$ are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$ and which consists of the following set of constraints:

$$d_E(x, y) = 1, \tag{1}$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \tag{2}$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{3}$$

if $d_{A \cup U \cup V}(x, y), d_{B \cup U \cup W}(y, z), d_{C \cup V \cup W}(z, x) \in \{0, 1\}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) = \frac{3}{2}, \tag{4}$$

if $d_{A \cup U \cup V}(x, y) = \frac{1}{2}$ or $d_{B \cup U \cup W}(y, z) = \frac{1}{2}$ or $d_{C \cup V \cup W}(z, x) = \frac{1}{2}$, for all triples of alternatives x, y, z and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy the following conditions:

$A \neq \emptyset$ only if there exists $\mathbf{p} \in \Omega$ such that $x\mathbf{p}z\mathbf{p}y$,
$B \neq \emptyset$ only if there exists $\mathbf{p} \in \Omega$ such that $y\mathbf{p}x\mathbf{p}z$,
$C \neq \emptyset$ only if there exists $\mathbf{p} \in \Omega$ such that $z\mathbf{p}y\mathbf{p}x$,
$U \neq \emptyset$ only if there exists $\mathbf{p} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$,
$V \neq \emptyset$ only if there exists $\mathbf{p} \in \Omega$ such that $z\mathbf{p}x\mathbf{p}y$,
$W \neq \emptyset$ only if there exists $\mathbf{p} \in \Omega$ such that $y\mathbf{p}z\mathbf{p}x$.

Busetto et al. (2015) showed that this ternary program can be used to provide a general representation of ASWFs, with and without ties in the range. In particular they showed, in their Theorem 1, that there exists a one-to-one correspondence between the set of the solutions to IP1 on a given Ω and the set of all ASWFs on the same Ω. We now restate this fundamental theorem as it will be systematically used in the rest of the paper.

Theorem 1. *Consider a domain Ω. Given an ASWF on Ω, f, there exists a unique solution to IP1 on Ω, d, which corresponds to f. Given a solution* *to IP1 on Ω, d, there exists a unique ASWF on Ω, f, which corresponds to d.*

Kalai and Muller (1977) were the first who provided a complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs without ties. In their Theorem 2, they showed that there exists a nondictatorial ASWF without ties on Ω for $n \geq 2$ if and only if Ω satisfies some conditions of decomposability.

Busetto et al. (2018) used an amended version of the IP used by Sethuraman et al. (2003) to give a new and simpler proof of Theorem 2 in Kalai and Muller (1977). In order to obtain their characterization theorem, they needed to use a reformulation of the concept of decomposability proposed by Kalai and Muller (1977) which is based on the existence of two sets, $R_1, R_2 \in \mathcal{A}^2$, which satisfy the two conditions we are going to introduce.

Consider a set $R \subset \mathcal{A}^2$. Consider the following conditions on R.

Condition 1. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R$ implies that $(x, z) \in R$.

Condition 2. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be decomposable if and only if there exist two sets R_1 and R_2, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2.

On the basis of the reformulation of the concept of decomposability, Busetto et al. (2018) proved a characterization theorem which can be straightforwardly restated, in terms of IP1, in the following way.

Theorem 2. *There exists a nondictatorial binary solution to IP1 on Ω, d, for $n \geq 2$, if and only if Ω is decomposable.*

The previous result provides a simplified proof of Theorem 2 in Kalai and Muller (1977) since this theorem can be obtained as a corollary of Theorem 2.

Corollary 1. *There exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$, if and only if Ω is decomposable.*

Proof. It is an immediate consequence of Theorems 1 and 2. ∎

In order to obtain a characterization theorem for nondictatorial ASWFs with ties, Busetto et al. (2015) needed to restrict further the condition of decomposability, introducing a new notion which they defined as strict decomposability. We now provide the notion of strict decomposability.

Given a set $R \subset \mathcal{A}^2$, consider the following conditions on R.

Condition 3. There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R^*$ implies that $(x, z) \in R$.

Condition 4. There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples of alternatives x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R^*$ imply that $(x, z) \in R$, and $(x, y) \in R^*$ and $(y, z) \in R$ imply that $(x, z) \in R$.

A domain Ω is said to be strictly decomposable if and only if there exist four sets R_1, R_2, R_1^*, and R_2^*, with $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$; $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$; moreover, R_i, $i = 1, 2$, satisfies Condition 1; R_i and R_i^*, $i = 1, 2$, satisfy Condition 2; each pair (R_i, R_i^*), $i = 1, 2$, satisfies Conditions 3 and 4.

On the basis of the notion of strict decomposability, we can straightforwardly provide the following characterization of domains admitting nondictatorial ternary solutions to IP1, based on Theorem 4 Busetto et al. (2015).

Theorem 3. *There exists a nondictatorial ternary solution to IP1 on Ω, d, for $n \geq 2$, if and only if Ω is strictly decomposable.*

Busetto et al. (2015) then proved, in their Theorem 5, the following generalization of Theorem 2 in Kalai and Muller (1977) for ASWFs without ties, which we restate as a corollary of Theorem 3.

Corollary 2. *There exists a nondictatorial ASWF with ties on Ω, f, for $n \geq 2$, if and only if Ω is strictly decomposable.*

Proof. It is an immediate consequence of Theorems 1 and 3. ■

The following theorem restates Theorem 7 in Busetto et al. (2015) which shows that a strictly decomposable domain is always decomposable.

Theorem 4. *If a domain Ω is strictly decomposable, then it is decomposable.*

4 Simple majority rule and integer programming

In this section, we use integer programming to determine the domains on which the Simple Majority Rule (SMR) is an ASWF and we compare them with the domains admitting nondictatorial ASWFs analyzed in the previous section. We start with some preliminary definition.

A solution d to an IP on Ω is a SMR solution if for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $d_S(x, y) = 1$ if and only if $|S| > |S^c|$, $d_S(x, y) = \frac{1}{2}$ if and only if $|S| = |S^c|$, and $d_S(x, y) = 0$ if and only if $|S| < |S^c|$. It is immediate to verify that a SMR solution to an IP on Ω, d, is binary if and only if n is odd and ternary if and only if n is even.

An ASWF on Ω, f, is said to be based on the SMR if it corresponds to a solution to IP1 on the same Ω, d, which is a SMR solution. It is immediate to verify that an ASWF on Ω, f, based on the SMR is nondictatorial without ties if and only if n is odd and nondictatorial with ties if and only if n is even.

We now restate a theorem, proved by Sethuraman et al. (2003), which is an integer programming version of a result showed by Sen (1966). The result is based on the following domain restriction.

A domain Ω is said to contain a Condorcet triple if there are a triple x, y, z and $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \Omega$ such that $x\mathbf{p}_1 y\mathbf{p}_1 z$, $y\mathbf{p}_2 z\mathbf{p}_2 x$, and $z\mathbf{p}_3 x\mathbf{p}_3 y$.

We can now restate Theorem 5 in Sethuraman et al. (2003)

Theorem 5. *Let n be odd. There exists a SMR binary solution to IP1 on Ω, d, if and only if Ω does not contain a Condorcet triple.*

Proof. It follows by adapting, *mutatis mutandis*, the proof of Theorem 5 in Sethuraman et al. (2003) to IP1. ■

We can then easily derive Theorem 1 in Sen (1966) as a corollary to Theorem 5.

Corollary 3. *Let n be odd. There exists an ASWF on Ω, f, based on the SMR if and only if Ω does not contain a Condorcet triple.*

Proof. It is an immediate consequence of Theorems 1 and 5. ■

Theorems 5 holds when n is odd. We shall now use integer programming to state and prove a new theorem which characterizes the domains providing a SMR solution to IP1 when n is even. The result is based on the following domain restriction, which combines Conditions B'' and C introduced by Inada (1969).

A domain Ω is said to be echoic with antagonistic preferences if, for all triples x, y, z, $\mathbf{p} \in \Omega$ and $x\mathbf{p}y\mathbf{p}z$ imply that only one of the following cases holds: (i) $x\mathbf{q}y\mathbf{q}z$ or $x\mathbf{q}z\mathbf{q}y$; (ii) $x\mathbf{q}y\mathbf{q}z$ or $y\mathbf{q}x\mathbf{q}z$; (iii) $x\mathbf{q}y\mathbf{q}z$ or $z\mathbf{q}y\mathbf{q}x$, for each $\mathbf{q} \in \Omega$ with $\mathbf{q} \neq \mathbf{p}$.[1]

We can now state and prove our new characterization theorem.

Theorem 6. *Let n be even. There exists a SMR ternary solution to IP1 on Ω, d, if and only if Ω is echoic with antagonistic preferences.*

Proof. Let n be even. Suppose that there exists a SMR ternary solution to IP1 on Ω, d. Suppose that Ω is not echoic with antagonistic preferences. Consider a triple x, y, z and suppose that $\mathbf{p} \in \Omega$ and $x\mathbf{p}y\mathbf{p}z$. Consider the case where $|U| = \frac{n}{2} = |V|$. Then, we have that

$$d_{U \cup V}(x, y) + d_U(y, z) + d_V(z, x) = 2,$$

as $d_{U \cup V}(x, y) = 1$, $d_U(y, z) = \frac{1}{2}$, and $d_V(z, x) = \frac{1}{2}$, contradicting (4). Consider the case where $|U| = \frac{n}{2} = |W|$. Then, by using *mutatis mutandis* the above argument, it follows that d contradicts (4). Consider the case where $|A| = \frac{n}{2} = |B|$. Then, we have that

$$d_A(x, y) + d_B(y, z) + d_{\emptyset}(z, x) = 1,$$

as $d_A(x, y) = \frac{1}{2}$, $d_B(y, z) = \frac{1}{2}$, and $d_{\emptyset}(z, x) = 0$, contradicting (4). Consider the case where $|A| = \frac{n}{2} = |C|$ or the case where $|B| = \frac{n}{2} = |C|$. Then, by using *mutatis mutandis* the above argument, it follows that d contradicts (4). We have exhausted all possible cases. Therefore, Ω must be echoic with antagonistic preferences. Conversely, suppose that Ω is echoic with antagonistic preferences. Determine d as follows. For each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $d_S(x, y) = 1$ if and only if $|S| > |S^c|$, $d_S(x, y) = \frac{1}{2}$ if and only if $|S| = |S^c|$, and $d_S(x, y) = 0$ if and only if $|S| < |S^c|$. Then, it is straightforward to verify that d satisfies (1) and (2). Consider a triple x, y, z and suppose that $\mathbf{p} \in \Omega$ and $x\mathbf{p}y\mathbf{p}z$. Suppose that $\mathbf{q} \in \Omega$ with $x\mathbf{q}\mathbf{y}\mathbf{q}z$. Then, we have that $A = \emptyset$, $B = \emptyset$, $C = \emptyset$, $V = \emptyset$, $W = \emptyset$. But then, we also have that $d_U(x, y) = d_E(x, y) = d_U(y, z) = d_E(y, z) = 1$ and $d_{\emptyset}(z, x) = 0$. Thus, d satisfies (3). Suppose that $\mathbf{q} \in \Omega$ with $x\mathbf{q}z\mathbf{q}y$. Then, we have that $B = \emptyset$, $C = \emptyset$, $V = \emptyset$, $W = \emptyset$. But then, we also have that $d_{A \cup U}(x, y) = d_E(x, y) = 1$ and $d_{\emptyset}(z, x) = 0$. Suppose that $d_U(y, z) = 0$ or

[1]This definition is based on the definitions of echoic and antagonistic preferences in Inada (1969).

$d_U(y,z) = 1$. Then, we have that

$$d_{A\cup U}(x,y) + d_U(y,z) + d_{\emptyset}(z,x) \leq 2,$$

as $d_{A\cup U}(x,y) = d_E(x,y) = 1$ and $d_{\emptyset}(z,x) = 0$. But then, d satisfies (3). Suppose that $d_U(y,z) = \frac{1}{2}$. Then, we have that

$$d_{A\cup U}(x,y) + d_U(y,z) + d_{\emptyset}(z,x) = \frac{3}{2},$$

as $d_{A\cup U}(x,y) = d_E(x,y) = 1$ and $d_{\emptyset}(z,x) = 0$. But then, d satisfies (4). Suppose that $\mathbf{q} \in \Omega$ with $y\mathbf{q}x\mathbf{q}z$. Then, by using *mutatis mutandis* the above argument, it follows that d satisfies (3) or (4). Suppose that $\mathbf{q} \in \Omega$ with $z\mathbf{q}y\mathbf{q}x$. Then, we have that $A = \emptyset$, $B = \emptyset$, $V = \emptyset$, $W = \emptyset$. But then, we also have that $C = U^c$. Suppose that $|U| > |U^c|$. Then, we have that

$$d_U(x,y) + d_U(y,z) + d_C(z,x) = 2,$$

as $d_U(x,y) = 1$, $d_U(y,z) = 1$, and $d_C(z,x) = 0$. Suppose that $|U| < |U^c|$. Then, we have that

$$d_U(x,y) + d_U(y,z) + d_C(z,x) < 2,$$

as $d_U(x,y) = 0$, $d_U(y,z) = 0$, and $d_C(z,x) = 1$. But then, d satisfies (3). Suppose that $|U| = |U^c|$. Then, we have that

$$d_U(x,y) + d_U(y,z) + d_C(z,x) = \frac{3}{2},$$

as $d_U(x,y) = \frac{1}{2}$, $d_U(y,z) = \frac{1}{2}$, and $d_C(z,x) = \frac{1}{2}$. But then, d satisfies (4). We have exhausted all possible cases. Therefore, d is a SMR ternary solution to IP1 on Ω. Hence, there exists a SMR ternary solution to IP1 on Ω, d, if and only if Ω is echoic with antagonistic preferences. ■

We can straightforwardly obtain the following corollary.

Corollary 4. *Let n be even. There exists an ASWF on Ω, f, based on the SMR if and only if Ω is echoic with antagonistic preferences.*

Proof. It is an immediate consequence of Theorems 1 and 6. ■

We now investigate the relationships among the domains admitting non-dictatorial ASWFs and those admitting an ASWF based on the SMR. We first consider the relationship between a domain which does not contain a Condorcet triple and a decomposable domain.

Proposition 1. *If Ω does not contain a Condorcet triple, then it is decomposable.*

Proof. Suppose that Ω does not contain a Condorcet triple. Let n be odd. Then, there exists a SMR binary solution to IP1 on Ω, d, by Theorem 5. Hence, Ω is decomposable, by Theorem 2. ∎

The next proposition shows that a domain which is echoic with antagonistic preferences is strictly decomposable.

Proposition 2. *In Ω is echoic with antagonistic preferences, then it is strictly decomposable.*

Proof. Suppose that Ω echoic with antagonistic preferences. Let n be even. Then, there exists a SMR ternary solution to IP1 Ω, d, by Theorem 6. Hence, Ω is strictly decomposable, by Theorem 3. ∎

The following example shows that the converse of Propositions 1 and 2 does not hold.

Example. *Let $\mathcal{A} = \{a, b, c\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}b\mathbf{p}c, b\mathbf{p}c\mathbf{p}a, c\mathbf{p}a\mathbf{p}b\}$. Then, Ω is strictly decomposable and decomposable but it contains a Condorcet triple and it is not echoic with antagonistic preferences.*

Proof. Let $R_1 = \{(a,c),(b,a),(c,b)\}$, $R_2 = \emptyset$, $R_1^* = \{(a,b),(b,c),(c,a)\}$, $R_2^* = \{(b,a),(c,b),(a,c)\}$. We have $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1, 2$. Moreover, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(x,y) \notin R_1^*$ and $(y,x) \notin R_2$; $(x,y) \in R_1^*$ if and only if $(y,x) \in R_2^*$. R_1 vacuously satisfies Conditions 1 and 2. R_1^* vacuously satisfies Condition 2. Moreover, the pair (R_1, R_1^*) satisfies Condition 3, as $(x,y) \in R_1^*$ and $(x,z) \in R_1$, and it vacuously satisfies Condition 4. R_2 vacuously satisfies Conditions 1 and 2. R_2^* vacuously satisfies Condition 2. Moreover, the pair (R_2, R_2^*) vacuously satisfies Conditions 3 and 4. Therefore, Ω is strictly decomposable. Moreover, Ω is decomposable, by Theorem 4. Nevertheless, Ω contains a Condorcet triple and it is not echoic with antagonistic preferences as it admits three ways of ordering the triple a, b, c. ∎

Our last proposition shows that the set of domains admitting an ASWF with ties based on the SMR is a strict subset of the set of domains admitting an ASWF without ties based on the SMR.

Proposition 3. *The set of domains admitting an ASWF based on the SMR when n is even is a strict subset of the set of domains admitting an ASWF based on the SMR when n is odd.*

Proof. Suppose that Ω is a domain admitting an ASWF based on the SMR when n is even. Then, Ω is echoic with antagonistic preferences, by Corollary 4. But then, it is straightforward to verify that it does not contain a Condorcet triple. Therefore, Ω admits an ASWF based on the SMR when n is odd, by Corollary 3. Thus, the set of domains admitting an ASWF based on the SMR when n is even is a subset of the set of domains admitting an ASWF based on the SMR when n is odd. Let $\mathcal{A} = \{a, b, c\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}b\mathbf{p}c, b\mathbf{p}c\mathbf{p}a, a\mathbf{p}c\mathbf{p}b\}$. It is immediate to verify that Ω does not contain a Condorcet triple. Then, Ω is a domain admitting an ASWF based on the SMR when n is odd, by Corollary 3. Nevertheless, Ω is not echoic with antagonistic preferences as it admits three ways of ordering the triple a, b, c. Then, Ω does not admit a ASWF based on the SMR when n is even. Hence, the set of domains admitting an ASWF based on the SMR when n is even is a strict subset of the set of domains admitting an ASWF based on the SMR when n is odd. ■

5 Conclusion

In this paper, we have systematically used integer programming to restate the characterization of the domains admitting nondictatorial ASWFs with and without ties. We have applied the integer programming approach to the SMR, which provides the most basic example of a nondictatorial ASWF with and without ties. In particular, we have proved the main result of this paper which shows that, when number of agents is even, a necessary and sufficient condition for the SMR to be an ASWF is that it is defined on a domain which is echoic with antagonistic preferences.

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