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On the Foundation of Monopoly in Bilateral Exchange*

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Abstract

We consider the mixed version of a monopolistic two-commodity exchange economy where the monopolist, represented as an atom, holds one commodity, and the "small traders," represented by an atomless part, hold the other. We provide a foundation of the monopoly solution in this framework by formulating an explicit trading process. We show that, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, our monopoly solution coincides with that defined by Kats (1974). Moreover, we show that, if the aggregate demand of the atomless part is not only invertible but also differentiable, our monopoly solution has the geometric characterization proposed by Schydlowsky and Siamwalla (1966).

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1 Introduction

Schydlowsky and Siamwalla (1966) opened a line of research on monopoly in general equilibrium. Concerning pure exchange economies, they considered a bilateral exchange where one commodity is held by one trader behaving as a monopolist while the other is held by a "competitors' community." They gave a geometrical representation of the monopoly solution as the point of tangency between the monopolist's indifference curve and the offer curve of the competitors' community. Some years later, Kats (1974) analyzed a pure exchange economy where one trader behaves as a monopolist, "calling the game" and maximizing his utility, whereas all the other traders in the economy behave competitively. He claimed that the monopoly quantity-setting solution must correspond to the monopolist's most preferred commodity bundle compatible with the aggregate initial endowments and the offer curve of the competitive traders.

In this paper, we provide a foundation of the monopoly solution in the framework of bilateral exchange by formulating an explicit trading process: to the best of our knowledge, this is a first attempt in this direction.

We consider the mixed version of a monopolistic two-commodity exchange economy introduced by Shitovitz (1973) in his Example 1, in which one commodity is held only by the monopolist, represented as an atom, and the other in held only by small traders, represented by an atomless part. This framework can also be used to represent a finite exchange economy if the atomless part is split into a finite number of types with traders of the same type having the same endowments and preferences.

We assume that,m within this framework, a trading process works as follows. The monopolist acts strategically making a bid of the commodity he holds for the other commodity, while the atomless part behaves à la Walras. Given the monopolist's bid, prices adjust to equate the monopolist's bid to the aggregate net demands of the atomless part. Each trader belonging to the atomless part then obtains his Walrasian demand whereas monopolist's final holding is determined as the difference between his endowment and his bid, for the commodity he holds, and as the value of his bid in terms of relative prices, for the other commodity. We define a monopoly equilibrium as a strategy played by the monopolist, corresponding to a positive bid of the commodity he holds, which guarantees him to obtain, via the trading process described above, a most preferred final holding among those he can achieve through his bids.

The general framework proposed in this paper to define and analyse monopoly equilibrium in bilateral exchange can be simplified, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, and compared with the standard partial equilibrium analysis of monopoly. Indeed we show that, if this assumption holds, at an allocation corresponding to a monopoly equilibrium, the utility of the monopolist is maximal in the feasible (with respect to aggregate endowments) complement of the offer curve of the atomless part, thereby providing a foundation of the monopoly solution proposed by Kats (1974). Moreover, we show that, if the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, a monopoly equilibrium has the geometric characterization proposed by Schydlowsky and Siamwalla (1966). This result lies on a notion which has a counterpart in partial equilibrium analysis: the marginal revenue of the monopolist.

The paper is organized as follows. In Section 2, we introduce the mathematical model. In Section 3, we define the notion of a monopoly equilibrium. In Section 4, we characterize the monopoly equilibrium when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible. In Section 5, we draw some conclusions and we suggest some further lines of research.

2 Mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. Let T_0 denote the atomless part of T. We assume that $\mu(T_0) > 0$.[1] Moreover, we assume that $T \backslash T_0 = \{a\}$, i.e., the measure space $(T, \mathcal{T}, \mu)$ contains only one atom, the "monopolist." A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word "integrable" is to be understood in the sense of Lebesgue.

[1] The symbol 0 denotes the origin of R^2_+ as well as the real number zero: no confusion will result.

In the exchange economy, there are two different commodities. A commodity bundle is a point in R^2_+. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}$: $T \to R^2_+$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. *$\mathbf{w}^i(a) > 0$, $\mathbf{w}^j(a) = 0$ and $\mathbf{w}^i(t) = 0$, $\mathbf{w}^j(t) > 0$, for each $t \in T_0$, $i = 1$ or 2, $j = 1$ or 2, $i \neq j$.*

An allocation is an assignment $\mathbf{x}$ such that $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$. The preferences of each trader $t \in T$ are described by a utility function $u_t : R^2_+ \to R$, satisfying the following assumptions.

Assumption 2. *$u_t : R^2_+ \to R$ is continuous, strongly monotone, and strictly quasi-concave, for each $t \in T$.*

Let $\mathcal{B}$ denote the Borel σ-algebra of R^2_+. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. *$u : T \times R^2_+ \to R$, given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^2_+$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.*

In order to state a last assumption, we need a preliminary definition. We say that commodities i, j stand in relation Q if $\mathbf{w}^i(t) > 0$, for each $t \in T_0$, and there is a nonnull subset T^i of T_0 such that $u_t(\cdot)$ is differentiable, additively separable, i.e., $u_t(x) = v^i_t(x^i) + v^j_t(x^j)$, for each $x \in R^2_+$, and $\frac{dv^j_t(0)}{dx^j} = +\infty$, for each $t \in T^i$.[2] Then, our last assumption can be formulated as follows.

Assumption 4. *Commodities i, j stand in relation Q.*

A price vector is a nonnull vector $p \in R^2_+$. Let $\mathbf{X}^0 : T_0 \times R^2_{++} \to \mathcal{P}(R^2_+)$ be a correspondence such that, for each $t \in T_0$ and for each $p \in R^2_{++}$, $\mathbf{X}^0(t, p) = \operatorname{argmax}\{u(x) : x \in R^2_+ \text{ and } px \leq p\mathbf{w}(t)\}$. For each $p \in R^2_{++}$, let $\int_{T_0} \mathbf{X}^0(t, p)\, d\mu = \{\int_{T_0} \mathbf{x}(t, p)\, d\mu : \mathbf{x}(\cdot, p) \text{ is integrable and } \mathbf{x}(t, p) \in \mathbf{X}^0(t, p), \text{ for each } t \in T_0\}$. Since the correspondence $\mathbf{X}^0(t, \cdot)$ is nonempty and single-valued, by Assumption 2, it is possible to define the Walrasian demand of traders in the atomless part as the function $\mathbf{x}^0 : T_0 \times R^2_{++} \to R^2_+$ such that $\mathbf{X}^0(t, p) = \{\mathbf{x}^0(t, p)\}$, for each $t \in T_0$ and for each $p \in R^2_{++}$. We can now state and show the following proposition.

[2] In this definition, differentiability means continuous differentiability and is to be understood as including the case of infinite partial derivatives along the boundary of the consumption set (for a discussion of this case, see, for instance, Kreps (2012), p. 58).

Proposition 1. *Under Assumptions 1, 2, and 3, the function* $\mathbf{x}^0(\cdot, p)$ *is integrable and* $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ *for each* $p \in R^2_{++}$.

Proof. Let $p \in R^l_{++}$. Then, the graph of the correspondence $\mathbf{X}(\cdot, p)$, $\{(t,x) : x \in \mathbf{X}(\cdot,p)\}$, is a subset of $\mathcal{T} \bigotimes \mathcal{B}$, by the same argument as that used by Busetto et al. (2011) (see the proof of their Proposition). But then, by the measurable choice theorem in Aumann (1969), there exists a measurable function $\bar{\mathbf{x}}(\cdot, p)$ such that, $\bar{\mathbf{x}}(t,p) \in \mathbf{X}(t,p)$, for each $t \in T_0$, which is also integrable as $\bar{\mathbf{x}}^j(t,p) \leq \frac{\sum_{i=1}^l p^i \mathbf{w}^i(t)}{p^j}$, $j = 1, 2$, for each $t \in T_0$. We must have that $\mathbf{x}^0(\cdot,p) = \bar{\mathbf{x}}(\cdot,p)$ as $\mathbf{X}^0(t,p) = \{\mathbf{x}^0(t,p)\}$, for each $t \in T_0$. Hence, the function $\mathbf{x}^0(\cdot,p)$ is integrable and $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$, for each $p \in R^2_{++}$. ∎

3 Monopoly equilibrium

We now provide the definition of a monopoly equilibrium in the bilateral exchange model introduced in the previous section. Let $\mathbf{E}(a) = \{(e_{ij}) \in R^4_+ : \sum_{j=1}^2 e_{ij} \leq \mathbf{w}^i(a),\ i = 1, 2\}$ denote the strategy set of atom a. We denote by $e \in \mathbf{E}(a)$ a strategy of atom a, where e_{ij}, $i, j = 1, 2$, represents the amount of commodity i that atom a offers in exchange for commodity j. Moreover, we denote by E the matrix corresponding to a strategy $e \in \mathbf{E}(a)$.

We then provide the following definitions.

Definition 1. *A square matrix C is said to be triangular if $c_{ij} = 0$ whenever $i > j$ or $c_{ij} = 0$ whenever $i < j$.*

Definition 2. *Given a strategy $e \in \mathbf{E}(a)$, a price vector p is said to be market clearing if*

$$p \in R^2_{++},\ \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu + \sum_{i=1}^2 e_{ij}\mu(a)\frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^2 e_{ji}\mu(a), \quad (1)$$

$j = 1, 2$.

The following proposition shows that market clearing price vectors can be normalized.

Proposition 2. *Under Assumptions 1, 2, and 3, if p is a market clearing price vector, then αp, with $\alpha > 0$, is also a market clearing price vector.*

Proof. It straightforwardly follows from homogeneity of degree zero of the function $\mathbf{x}^0(t, \cdot)$, for each $t \in T_0$, and from (1). ∎

Henceforth, we say that a price vector p is normalized if $p \in \Delta$ where $\Delta = \{p \in R^2_+ : \sum_{i=1}^2 p^i = 1\}$. Moreover, we denote by $\partial\Delta$ the boundary of the unit simplex Δ.

The next proposition shows that the two equations in (1) are not independent.

Proposition 3. *Under Assumptions 1, 2, and 3, given a strategy $e \in \mathbf{E}(a)$, a price vector $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$.*

Proof. Let a strategy $e \in \mathbf{E}(a)$ be given. Suppose, without loss of generality, that $\mathbf{w}^1(a) > 0$. Let $p \in \Delta \setminus \partial\Delta$ be a price vector. Suppose that p is market clearing for $j = 1$. Then, (1) reduces to

$$\int_{T_0} \mathbf{x}^{01}(t,p) = e_{12}\mu(a).$$

We have that

$$p^1 \int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu + p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

as $p^1\mathbf{x}^{01}(t,p) + p^2\mathbf{x}^{02}(t,p) = p^2\mathbf{w}^2(t)$, by Assumption 2, for each $t \in T_0$. Then, we have that

$$\int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) + e_{12}\mu(a)\frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

Therefore, p is market clearing for $j = 2$. Suppose now that (1) is satisfied for $j = 2$. Then, (1) reduces to

$$\int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) + e_{12}\mu(a)\frac{p^1}{p^2} = \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

But then, we have that

$$p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) + p^1 e_{12}\mu(a) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

On the other hand, we know from the previous argument that

$$p^1 \int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu + p^2 \int_{T_0} \mathbf{x}^{02}\, d\mu(t,p) = p^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu.$$

Then, we obtain that

$$\int_{T_0} \mathbf{x}^{01}(t,p) = e_{12}\mu(a).$$

Therefore, p is market clearing for $j = 1$. Hence, $p \in \Delta \setminus \partial\Delta$ is market clearing for $j = 1$ if and only if it is market clearing for $j = 2$. ■

To prove the next proposition, according to Debreu (1982) we let $|x| = \sum_{i=1}^{2} |x^i|$, for each $x \in R^2_+$, and $d[0, V] = \inf_{x \in V} |x|$, for each $V \subset R^2_+$. The proposition is based on Property (iv) of the aggregate demand of an atomless set of traders established by Debreu (1982), p. 728.

Proposition 4. *Under Assumptions 1, 2, and 3, let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, and which converges to a normalized price vector $\bar{p}$. If $\bar{p}^i = 0$ and $\mathbf{w}^i(a) > 0$, then the sequence $\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$.*

Proof. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$. Suppose, without loss of generality, that $\bar{p}^1 = 0$ and $\mathbf{w}^1(a) > 0$. Then, we have that $\bar{p}^2 = 1$. But then, the sequence $\{d[0, \mathbf{X}^0(t, p^n)]\}$ diverges to $+\infty$ as $\bar{p}^2\mathbf{w}^2(t) > 0$, for each $t \in T_0$, by Lemma 4 in Debreu (1982), p. 721. Therefore, the sequence $\{d[0, \int_{T_0} \mathbf{X}^0(t, p^n)\, d\mu]\}$ diverges to $+\infty$, by the argument used in the proof of Property (iv) in Debreu (1982), p. 728. This implies that the sequence $\sum_{i=1}^{2}\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$ as $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Suppose that the sequence $\{\int_{T_0} \mathbf{x}^{02}(t, p^n)\, d\mu\}$ diverges to $+\infty$. Then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{02}(t, p^n)\, d\mu > \int_{T_0} \mathbf{w}^2(t)\, d\mu$, for each $n \geq n_0$. But we have that $\mathbf{x}^{02}(t,p) \leq \mathbf{w}^2(t)$, for each $t \in T_0$ and for each $p \in \Delta \setminus \partial\Delta$, a contradiction. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t, p^n)\, d\mu\}$ diverges to $+\infty$. Hence, the sequence $\{\int_{T_0} \mathbf{x}^{0i}(t, p^n)\, d\mu\}$ diverges to $+\infty$ whenever $\bar{p}^i = 0$ and $\mathbf{w}^i(a) > 0$. ■

The following proposition provides a necessary and sufficient condition for the existence of a market clearing price vector.

Proposition 5. *Under Assumptions 1, 2, 3, and 4, given a strategy $e \in \mathbf{E}(a)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular.*

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(a) > 0$ and let $e \in \mathbf{E}(a)$ be a strategy. Suppose that there exists a market clearing price vector $p \in$

$\Delta \setminus \partial\Delta$ and that the matrix E is not triangular. Then, it must be that $e_{12} = 0$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t,p)\, d\mu = 0$ as $\mu(T^2) > 0$, by (1). Consider a trader $\tau \in T^2$. We have that $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^1} = +\infty$ as 2 and 1 stand in the relation Q and $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^1} \leq \lambda \hat{p}^1$, by the necessary conditions of the Kuhn-Tucker theorem. Moreover, it must be that $\mathbf{x}^{02}(\tau,p) = \mathbf{w}^2(\tau) > 0$ as $u_\tau(\cdot)$ is strongly monotone, by Assumption 2, and $p\mathbf{w}(\tau) > 0$. Then, $\frac{\partial u_\tau(\mathbf{x}^0(\tau,p))}{\partial x^2} = \lambda p^2$, by the necessary conditions of the Kuhn-Tucker theorem. But then, $\frac{\partial u_\tau(\hat{\mathbf{x}}(\tau))}{\partial x^2} = +\infty$ as $\lambda = +\infty$, contradicting the assumption that $u_\tau(\cdot)$ is continuously differentiable. Therefore, the matrix E must be triangular. Suppose now that E is triangular. Then, it must be that $e_{12} > 0$. Let $\{p^n\}$ be a sequence of normalized price vectors such that $p^n \in \Delta \setminus \partial\Delta$, for each $n = 1, 2, \ldots$, which converges to a normalized price vector $\bar{p}$ such that $\bar{p}^1 = 0$. Then, the sequence $\{\int_{T_0} \mathbf{x}^{01}(t,p^n)\, d\mu\}$ diverges to $+\infty$, by Proposition 4. But then, there exists an n_0 such that $\int_{T_0} \mathbf{x}^{01}(t,p^n)\, d\mu > e_{12}\mu(a)$, for each $n \geq n_0$. Therefore, we have that $\int_{T_0} \mathbf{x}^{01}(t,p^{n_0})\, d\mu > e_{12}\mu(a)$. Let $q \in \Delta \setminus \partial\Delta$ be a price vector such that $\frac{q^2 \int_{T_0} \mathbf{w}^2(t)\, d\mu}{q^1} = e_{12}\mu(a)$. Consider first the case where $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu = e_{12}\mu(a)$. Then, q is market clearing as it is market clearing for $j = 1$, by Proposition 3. Consider now the case where $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu \neq e_{12}\mu(a)$. Then, it must be that $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu < e_{12}\mu(a)$ as $\mathbf{x}^{01}(t,q) \leq \frac{q^2 \mathbf{w}^2(t)}{q^1}$, for each $t \in T_0$. But then, we have that $\int_{T_0} \mathbf{x}^{01}(t,q)\, d\mu < e_{12}\mu(a) < \int_{T_0} \mathbf{x}^{01}(t,p^{n_0})\, d\mu$. Let $O \subset \Delta \setminus \partial\Delta$ be a compact and convex set which contains p^{n_0} and q. Then, the correspondence $\int_{T_0} \mathbf{X}^0(t,\cdot)\, d\mu$ is upper hemicontinuous on O, by the argument used in the proof of Property (ii) in Debreu (1982), p. 728. But then, the function $\{\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu\}$ is continuous on O as $\int_{T_0} \mathbf{X}^0(t,p)\, d\mu = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$, for each $p \in \Delta \setminus \partial\Delta$, by Proposition 1. Therefore, there is a price vector $p^* \in \Delta \setminus \partial\Delta$ such that $\int_{T_0} \mathbf{x}^{01}(t,p^*)\, d\mu = e_{12}\mu(a)$, by the intermediate value theorem. Then, p^* is market clearing as it is market clearing for $j = 1$, by Proposition 3. Hence, given a strategy $e \in \mathbf{E}(a)$, there exists a market clearing price vector $p \in \Delta \setminus \partial\Delta$ if and only if the matrix E is triangular. ■

We denote now by $\pi(e)$ a correspondence which associates, with each strategy $e \in \mathbf{E}(a)$, the set of price vectors p satisfying (1), if E is triangular, and is equal to $\{0\}$, otherwise. A price selection $p(e)$ is a function which associates, with each strategy selection $e \in \mathbf{E}(a)$, a price vector $p \in \pi(e)$.

Given a strategy $e \in \mathbf{E}(a)$ and a price vector p, consider the assignment determined as follows:

$$\begin{aligned}
\mathbf{x}^j(a,e,p) &= \mathbf{w}^j(a) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j},\ \text{if } p \in R^2_{++},\\
\mathbf{x}^j(a,e,p) &= \mathbf{w}^j(a),\ \text{otherwise},
\end{aligned}$$

$j = 1, 2$,

$$\begin{aligned}
\mathbf{x}^j(t,p) &= \mathbf{x}^{0j}(t,p),\ \text{if } p \in R^2_{++},\\
\mathbf{x}^j(t,p) &= \mathbf{w}^j(t),\ \text{otherwise},
\end{aligned}$$

$j = 1, 2$, for each $t \in T_0$.

Given a strategy $e \in \mathbf{E}(a)$ and a price selection $p(e)$, traders' final holdings are determined according to this rule and consequently expressed by the assignment

$$\mathbf{x}(a) = \mathbf{x}(a, e, p(e)),$$

$$\mathbf{x}(t) = \mathbf{x}(t, p(e)),$$

for each $t \in T_0$.

The next proposition shows that traders' final holdings are an allocation.

Proposition 6. *Under Assumptions 1, 2, 3, and 4, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$, the assignment $\mathbf{x}(a) = \mathbf{x}(a, e, p(e))$, $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$, is an allocation.*

Proof. Let a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$ be given. Suppose that E is not triangular. Then, we have that $\mathbf{x}(a) = \mathbf{x}(a, e, p(e)) = \mathbf{w}(a)$ and $\mathbf{x}(t) = \mathbf{x}(t, p(e)) = \mathbf{w}(t)$, for each $t \in T_0$ as $p(e) = 0$. Suppose that E is triangular. Then, we have that

$$\int_T \mathbf{x}^j(t)\, d\mu = (\mathbf{w}^j(a) - \sum_{i=1}^{2} e_{ji} + \sum_{i=1}^{2} e_{ij}\frac{p^i}{p^j})\mu(a) + \int_{T_0} \mathbf{x}^{0j}(t,p)\, d\mu = \int_T \mathbf{w}^j(t)\, d\mu,$$

$j = 1, 2$, as $p(e)$ is market clearing. Hence, given a price selection $p(\cdot)$ and a strategy $e \in \mathbf{E}(a)$, the assignment $\mathbf{x}(a) = \mathbf{x}(a, e, p(e))$, $\mathbf{x}(t) = \mathbf{x}(t, p(e))$, for each $t \in T_0$, is an allocation. ■

We can now provide the definition of a monopoly equilibrium.

Definition 3. *A strategy* $\tilde{e} \in \mathbf{E}(a)$ *such that* $\tilde{E}$ *is triangular is a monopoly equilibrium, with respect to a price selection* $p(\cdot)$*, if*

$$u_a(\mathbf{x}(a, \tilde{e}, p(\tilde{e})) \geq u_a(\mathbf{x}(a, e, p(e)),$$

for each $e \in \mathbf{E}(a)$*.*

A monopoly allocation is an allocation $\tilde{\mathbf{x}}$ such that $\tilde{\mathbf{x}}(a) = \mathbf{x}(a, \tilde{e}, p(\tilde{e}))$ and $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{e}))$, for each $t \in T_0$, where $\tilde{e}$ is a monopoly equilibrium.

4 Monopoly equilibrium and invertible aggregate demand

The analysis of the monopoly problem in bilateral exchange proposed in the previous sections can be simplified by introducing the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and compared, under this restriction, with the standard partial equilibrium analysis of monopoly.

The following proposition states a necessary and sufficient condition for the atomless part's aggregate demand to be invertible.

Proposition 7. *Under Assumptions 1, 2, 3, and 4, let* $\mathbf{w}^i(a) > 0$*. Then, the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible if and only if, for each* $x \in R_{++}$*, there is a unique* $p \in \Delta \setminus \partial\Delta$ *such that* $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$*.*

Proof. Let $\mathbf{w}^i(a) > 0$. Suppose that $\int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. Then, we have that $\int_{T^i} \mathbf{x}^{0i}(t, p)\, d\mu = 0$ as $\mu(T^i) > 0$ and the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of Proposition 5. But then, we have that $\int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu > 0$, for each $p \in \Delta \setminus \partial\Delta$. Therefore, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is restricted to the codomain R_{++}. For each $x \in R_{++}$, there exists at least one $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$, by the same argument used in the proof of Proposition 5. Then, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is onto as its range coincides with its codomain. Therefore, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible if and only if it is one-to-one. Hence, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible if and only, for each $x \in R_{++}$, there is a unique $p \in \Delta \setminus \partial\Delta$ such that $x = \int_{T_0} \mathbf{x}^{0i}(t, p)\, d\mu$. ■

Let $p^{0i}(\cdot)$ denote the inverse of the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$. The following proposition shows that when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, there exists a unique price selection.

Proposition 8. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(a) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, then there exists a unique price selection* $\mathring{p}(\cdot)$.

Proof. Suppose that $\mathbf{w}^i(a) > 0$ and that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible. Let $\mathring{p}(e)$ be a function which associates, with each strategy $e \in \mathbf{E}(a)$, the price vector $p = p^{0i}(e_{ij}\mu(a))$, if E is triangular, and is equal to $\{0\}$, otherwise. Then, $\mathring{p}(\cdot)$ is the unique price selection as $\pi(e) = \{\mathring{p}(e)\}$, for each $e \in E(a)$. ■

By analogy with partial analysis, $\mathring{p}(\cdot)$ can be called the inverse demand function of the monopolist. When the aggregate demand of the atomless part for the commodity held by the monopolist in invertible, the monopoly equilibrium can be reformulated as in Definition 3, with respect to monopolist's inverse function $\mathring{p}(\cdot)$.

Moreover, under this assumption, the monopoly equilibrium can be characterized by means of the notion of offer curve of the atomelss part, defined as set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$.

If $\mathbf{w}^i(a) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, we let $h(\cdot)$ be a function, defined on R_{++}, such that

$$p^i x^i + p^j x^j = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu,$$

where $p = p^{0i}(x^i)$ and $x^j = h(x^i)$.

The following proposition shows that the function $h(\cdot)$ represents the offer curve of the atomless part in the sense that its graph coincides with the atomless part's offer curve.

Proposition 9. *Under Assumptions 1, 2, 3, and 4, if* $\mathbf{w}^i(a) > 0$ *and the function* $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ *is invertible, then the graph of the function* $h(\cdot)$*, the set* $\{x \in R^2_+ : x^j = h(x^i)\}$*, coincides with the set* $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t, p)\, d\mu \text{ for some } p \in \Delta \setminus \partial\Delta\}$*, the offer curve of the atomless part.*

Proof. Suppose that $\mathbf{w}^i(a) > 0$ and that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible. Suppose that $\bar{x} \in \{x \in R^2_+ : x^j = h(x^i)\}$. Then, there is a unique price vector $\bar{p} = p^{0i}(\bar{x}^i)$ such that $\bar{x}^i = \int_{T_0} \mathbf{x}^{0i}(t, \bar{p})\, d\mu$, as the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. We have that

$$\bar{p}^i \int_{T_0} \mathbf{x}^{0i}(t,\bar{p})\, d\mu + \bar{p}^j \int_{T_0} \mathbf{x}^{0j}(t,\bar{p})\, d\mu = p^i \int_{T_0} \mathbf{w}^i(t)\, d\mu + p^j \int_{T_0} \mathbf{w}^j(t)\, d\mu,$$

by Walras' law. Then, it must be that $\bar{x}^j = \int_{T_0} \mathbf{x}^{0j}(t,\bar{p})\, d\mu$, where $\bar{x}^j = h(\bar{x}^i)$. But then, $\bar{x} \in \{x \in R^2 : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Therefore, $\{x \in R^2_+ : x^j = h(x^i)\} \subset \{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in R^2_{++}\}$. Suppose now that $\bar{x} \in \{x \in R^2 : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Let $\bar{p}$ be such that $\bar{x} = \int_{T_0} \mathbf{x}^0(t,\bar{p})\, d\mu$. Then, we have that $\bar{p} = p^{0i}(\bar{x}^i)$ as the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible. We have that

$$\bar{p}^i \bar{x}^i + \bar{p}^j \bar{x}^j = \bar{p}^i \int_{T_0} \mathbf{w}^i(t) + p^j \int_{T_0} \mathbf{w}^j(t),$$

by Walras' law. Then, we have that $\bar{x}^j = h(\bar{x}^i)$. But then, $\bar{x} \in \{x \in R^2_+ : x^j = h(x^i)\}$. Therefore, $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R^2_+ : x^j = h(x^i)\}$. Hence, the graph of the function $h(\cdot)$, the set $\{x \in R^2_+ : x^j = h(x^i)\}$, coincides with the set $\{x \in R^2_+ : x = \int_{T_0} \mathbf{x}^0(t,p)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, the offer curve of the atomless part. ■

A first characterization of a monopoly equilibrium can now be provided introducing the notion of feasible complement of the offer curve of the atomless part. It is defined as the set $\{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$.

The following proposition shows that, when the aggregate demand of the atomless part for the commodity held by the monopolist is invertible, the feasible complement of the atomless part's offer curve is a subset of the set of the monopolist's final holdings.

Proposition 10. *Under Assumptions 1, 2, 3, and 4, if $\mathbf{w}^i(a) > 0$ and the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible, then the feasible complement of the offer curve of the atomless part, the set $\{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R^2_+ : x = \mathbf{x}(a, e, \mathring{p}(e))$ for some $e \in E(a)\}$, the set of the final holdings of the monopolist.*

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(a) > 0$ and that $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Suppose that $\bar{x} \in \{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. Moreover, suppose that $\bar{x}^1 = \mathbf{w}^1(a)$.

Then, we have that $\int_{T_0} \mathbf{x}^{01}(t,p)\, d\mu = 0$, for some $p \in \Delta \setminus \partial\Delta$. But then, we have that $\int_{T^2} \mathbf{x}^{01}(t,p)\, d\mu = 0$ as $\mu(T^2) > 0$ and the necessary Kuhn-Tucker conditions lead, *mutatis mutandis*, to the same contradiction as in the proof of Proposition 5. Therefore, we must have that $0 \leq \bar{x}^1 < \mathbf{w}^1(a)$. Let $e \in E(a)$ be such that $\bar{e}_{12} = \mathbf{w}^1(a) - \bar{x}^1$ and let $\bar{p} = \mathring{p}(\bar{e})$. Then, we have that

$$\bar{x}^1\mu(a) + \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu = (\mathbf{w}^1(a) - \bar{e}_{12})\mu(a) + \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu = \mathbf{w}^1(a)\mu(a),$$

as $\bar{p} = \mathring{p}(\bar{e})$. Moreover, $\bar{p}$ is the unique price vector such that

$$(\mathbf{w}^1(a) - \bar{x}^1)\mu(a) = \int_{T_0} \mathbf{x}^{01}(t,\bar{p})\, d\mu,$$

as the function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Then, it must be that

$$\bar{x}^2\mu(a) + \int_{T_0} \mathbf{x}^{02}(t,\bar{p})\, d\mu = \int_{T_0} \mathbf{w}^2(t)\, d\mu,$$

by Proposition 3. But then, we have that

$$\bar{x}^2 = e_{12}\frac{\bar{p}^1}{\bar{p}^2},$$

as $\bar{p}$ is market clearing. Therefore, we conclude that

$$\bar{x} = \mathbf{x}(a,\bar{e},\bar{p}) = \mathbf{x}(a,\bar{e},\mathring{p}(\bar{e})).$$

Hence, the feasible complement of the offer curve of the atomless part, the set $\{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$, is a subset of the set $\{x \in R^2_+ : x = \mathbf{x}(a,e,\mathring{p}(e))$ for some $e \in E(a)\}$, the set of the final holdings of the monopolist. ■

Then, we can show the following corollary to Proposition 10. It establishes that, at a monopoly allocation, the utility of the monopolist is maximal on the feasible complement of the atomless part's offer curve and, consequently, it provides a foundation of the monopoly solution proposed by Kats (1974).

Corollary 1. *Under Assumptions 1, 2, 3, and 4, if $\mathbf{w}^i(a) > 0$, the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible, and $\tilde{e} \in E(a)$ is a monopoly equilibrium, then*

$u_a(\mathbf{x}(a, \tilde{e}, \mathring{p}(\tilde{e})))$ *is maximal in the set* $\{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ *for some* $p \in \Delta \setminus \partial\Delta\}$.

Proof. Suppose, without loss of generality, that $\mathbf{w}^1(a) > 0$ and that the function $\int_{T_0} \mathbf{x}^{01}(t,\cdot)\, d\mu$ is invertible. Let $\tilde{e} \in E(a)$ be a monopoly equilibrium. Let $\tilde{p} = \mathring{p}(\tilde{e})$. We have that

$$\begin{aligned} \mathbf{x}^1(a,\tilde{e},\tilde{p})\mu(a) + \int_{T_0} \mathbf{x}^{01}(t,\tilde{p})\, d\mu &= (\mathbf{w}^1(a) - \tilde{e}_{12})\mu(a) + \int_{T_0} \mathbf{x}^{01}(t,\tilde{p})\, d\mu \\ &= \mathbf{w}^1(a)\mu(a), \end{aligned}$$

and

$$\begin{aligned} \mathbf{x}^2(a,\tilde{e},\tilde{p})\mu(a) + \int_{T_0} \mathbf{x}^{02}(t,\tilde{p})\, d\mu &= \tilde{e}_{12}\mu(a)\frac{\tilde{p}^1}{\tilde{p}^2} + \int_{T_0} \mathbf{x}^{02}(t,\tilde{p})\, d\mu \\ &= \int_{T_0} \mathbf{w}^2(t)\, d\mu, \end{aligned}$$

as $\tilde{p}$ is market clearing. Then, we have shown that $\mathbf{x}(a, \tilde{e}, \mathring{p}(\tilde{e})) \in \{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$. But then, we have that $u_a(\mathbf{x}(a, \tilde{e}, \mathring{p}(\tilde{e})))$ is maximal in the set $\{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\}$ as $u_a(\mathbf{x}(a, \tilde{e}, \mathring{p}(\tilde{e})) \geq u_a(\mathbf{x}(a, e, \mathring{p}(e))$, for each $e \in \mathbf{E}(a)$, and $\{x \in R^2_+ : x\mu(a) + \int_{T_0} \mathbf{x}^0(t,p)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$ for some $p \in \Delta \setminus \partial\Delta\} \subset \{x \in R^2_+ : x = \mathbf{x}(a, e, \mathring{p}(e))$ for some $e \in E(a)\}$, by Proposition 10. ■

We show now that, under the assumption that the aggregate demand of the atomless part for the commodity held by the monopolist is not only invertible but also differentiable, the monopoly equilibrium introduced in Definition 3 has also the geometric characterization previously proposed by Schydlowsky and Siamwalla (1966): at a strictly positive monopoly allocation the monopolist's indifference curve is tangent to the atomless part's offer curve.[3] We do it by introducing in our general framework a notion which has a counterpart in partial equilibrium analysis: the marginal revenue of the monopolist. In the rest of this section, to save in notation but with some abuse, given a price vector $(p^i, p^j) \in \Delta \setminus \partial\Delta$, we denote by p the scalar $p = \frac{p^i}{p^j}$, whenever $\mathbf{w}^i(a) > 0$. Suppose that $\mathbf{w}^i(a) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t,\cdot)\, d\mu$ is invertible, and that the function $p^{0i}(\cdot)$ is differentiable.

[3]This characterization of the monopoly equilibrium has been diffusely reproposed in standard textbooks in microeconomics (see, for instance, Varian (2014) p. 619, among others).

Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and we have that $\frac{d\mathring{p}(e)}{de_{ij}} = \frac{dp^{0i}(e_{ij}\mu(a))}{dx^i}\mu(a)$, at each $e \in E(a)$ such that E is triangular, by Proposition 8. In this context, the revenue of the monopolist can be defined as $\mathring{p}(e)e_{ij}$ and his marginal revenue as $\frac{d\mathring{p}(e)}{de_{ij}}e_{ij} + \mathring{p}(e)$, for each $e \in E(a)$ such that E is triangular.

In order to provide the geometric characterization of a monopoly equilibrium, we need to introduce also the following assumption.

Assumption 5. *$u_a : R^2_+ \to R$ is differentiable.*

Then, the next proposition provides a formal foundation to the geometric characterization of the monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966). Indeed, it establishes that, at an interior monopoly solution, the slope of the monopolist's indifference curve and the slope of the atomless part's offer curve are both equal to the opposite of the monopolist's marginal revenue, thereby showing the tangency of those two curves.

Proposition 11. *Under Assumptions 1, 2, 3, 4, and 5, if $\mathbf{w}^i(a) > 0$, the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible, the function $p^{0i}(\cdot)$ is differentiable, and $\tilde{e} \in E(a)$ is a monopoly equilibrium such that $\tilde{e} < \mathbf{w}^i(a)$, then*

$$-\frac{\frac{\partial u_a(\tilde{\mathbf{x}}(a)}{\partial x^i}}{\frac{\partial u_a(\tilde{\mathbf{x}}(a))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i},$$

where $\tilde{\mathbf{x}}$ is the monopoly allocation corresponding to $\tilde{e}$.

Proof. Suppose that $\mathbf{w}^i(a) > 0$, that the function $\int_{T_0} \mathbf{x}^{0i}(t, \cdot)\, d\mu$ is invertible and that the function $p^{0i}(\cdot)$ is differentiable. Let $\tilde{e} \in E(a)$ be a monopoly equilibrium such that $\tilde{e} < \mathbf{w}^i(a)$ and let $\tilde{\mathbf{x}}$ be the corresponding monopoly allocation. Then, $\mathring{p}(\cdot)$, the inverse demand function of the monopolist, is differentiable and the necessary Kuhn-Tucker conditions imply that

$$-\frac{\partial u_a(\tilde{\mathbf{x}}(a))}{\partial x^i} + \frac{\partial u_a(\tilde{\mathbf{x}}(a))}{\partial x^j}\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = 0.$$

Then, we have that

$$-\frac{\frac{\partial u_a(\tilde{\mathbf{x}}(a)}{\partial x^i}}{\frac{\partial u_a(\tilde{\mathbf{x}}(a))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right).$$

Moreover, we have that

$$h(x^i) = -p^{0i}(x^i)x^i + \int_{T_0} \mathbf{w}^j(t)\, d\mu.$$

Differentiating the function $h(\cdot)$, we obtain

$$\frac{dh(x^i)}{x^i} = -\left(\frac{dp^{0i}(x^i)}{dx^i}x^i + p^{0i}(x^i)\right).$$

At the monopoly allocation $\tilde{\mathbf{x}}$, we have that

$$\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e}) = \frac{dp^{0i}(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i} \int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu + p^{0i}(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu),$$

as $\frac{d\mathring{p}(\tilde{e})}{de_{ij}} = \frac{dp^{0i}(\tilde{e}_{ij}\mu(a))}{dx^i}\mu(a)$ and $\tilde{e}_{12}\mu(a) = \int_{T_0} \mathbf{x}^{0i}(t, p(\tilde{e}))\, d\mu$. Hence, we have that

$$-\frac{\frac{\partial u_a(\tilde{\mathbf{x}}(a)}{\partial x^i}}{\frac{\partial u_a(\tilde{\mathbf{x}}(a))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}.$$

■

Finally, We provide an example of the geometric characterization of a monopoly equilibrium proposed by Schydlowsky and Siamwalla (1966).

Example. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, 4, and 5. $T_0 = [0, 1]$, $T \setminus T_0 = \{a\}$, T_0 is taken with Lebesgue measure, $\mu(2) = 1$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = \sqrt{x}_1 + x_2$, for each $t \in T_0$, $\mathbf{w}(a) = (1, 0)$, $u_a(x) = \frac{1}{2}x_1 + \sqrt{x}_2$. Then, there is a unique monopoly equilibrium $\tilde{e} \in E(a)$ at which the slope of the indifference curve of the monopolist is equal to the opposite of his marginal revenue, which, in turn, is equal to the slope of the function which represents the offer curve of the atomless part.*

Proof. The unique monopoly equilibrium is the strategy $\tilde{e}$ such that $\tilde{e}_{12} = \frac{1}{4}$, $\mathring{p}(\tilde{e}) = 1$, $\tilde{\mathbf{x}}(t) = (\frac{1}{4}, \frac{3}{4})$, for each $t \in T_0$, and $\tilde{\mathbf{x}}(a) = (\frac{3}{4}, \frac{1}{4})$. Moreover, we have that $x^2 = h(x^1) = -\frac{\sqrt{x}^1}{2} + 1$ and

$$-\frac{\frac{\partial u_a(\tilde{\mathbf{x}}(a)}{\partial x^i}}{\frac{\partial u_a(\tilde{\mathbf{x}}(a))}{\partial x^j}} = -\left(\frac{d\mathring{p}(\tilde{e})}{de_{ij}}\tilde{e}_{ij} + \mathring{p}(\tilde{e})\right) = -\frac{1}{2} = \frac{dh(\int_{T_0} \tilde{\mathbf{x}}^i(t)\, d\mu)}{dx^i}.$$

■

5 Conclusion

In this paper, we have introduced the mixed version of a monopolistic two-commodity exchange economy and, in this framework, we have provided a foundation of the monopoly solution by formalizing an explicit trading process. We have first shown that, under the assumption that the aggregate demand of the atomless part of the economy for the commodity held by the monopolist is invertible, our monopoly solution coincides with that proposed by Kats (1974). We have then shown that, if the aggregate demand of the atomless part for the commodity held by the monopolist is invertible and differentiable, our monopoly solution has the geometric characterization proposed by Schydlowsky and Siamwalla (1966). We have provided an example of this configuration. We leave for further research the analysis of the existence and optimality properties of the concept of monopoly equilibrium founded in this paper.

Kats (1974), in his final remarks (see p. 31), raised the question of the relationship between monopoly equilibrium and cooperative game theory. He formalized a monopolistic market game based on the notion of a monopolistic quasi-core and referred that to Shitovitz (1973) as the only other contribution which had addressed a similar issue, using cooperative game theory. Shitovitz (1973) actually showed, in his Example 1, that, in the mixed version of a monopolistic two-commodity exchange economy, the set of allocations in the core does not coincide with the set of Walrasian allocations. This example raised the question whether the core solution to monopolistic market games is "advantageous" or "disadvantageous" for the monopolist (see Aumann (1973), Drèze et al. (1977), Greenberg and Shitovitz (1977), among others). The same issue could be analysed using our monopoly equilibrium solution.

Sadanand (1988) studied the mixed version of a monopolistic two-commodity exchange economy, in a noncooperative framework, using strategic market games à la Shapley and Shubik (see Giraud (2003) for a survey of this literature). He first confirmed the negative result on the existence of a Cournot-Nash equilibrium with trade for a monopolistic strategic market game in a one-stage setting, already discussed by Okuno et al. (1980) (see p. 24). He then recognized the two stage-flavor of monopoly equilibrium, when the monopolist is a price-maker. In a further step of our research, borrowing from the analysis of oligopoly in mixed exchange economies developed by Busetto et al. (2008), we propose to study a two-stage monopolistic strategic market game where a quantity-setting monopolist moves first and the atomless part moves in the second stage, after observing the moves on the monopolist in the first stage. This would provide a noncooperative foundation of our monopoly equilibrium.

References

[1] Aumann R.J. (1969), "Measurable utility and measurable choice theory," in Guilbaud G.T. (ed), *La Décision*, Editions du Centre National de la Recherche Scientifique, Paris.

[2] Aumann R.J. (1973), "Disadvantageous monopolies," *Journal of Economic Theory* **6**, 1-11.

[3] Busetto F., Codognato G., Ghosal S. (2008), "Cournot-Walras equilibrium as a subgame perfect equilibrium," *International Journal of Game Theory* **37**, 371-386.

[4] Debreu G. (1982), "Existence of competitive equilibrium," in Arrow K.J., Intriligator H.D. (eds), *Handbook of mathematical economics*, Elsevier, Amsterdam.

[5] Drèze J.H:, Gabszewicz J.J., Postlewaite A. (1977), "Disadvantageous monopolies and disadvantageous endowments," *Journal of Economic Theory* **16**, 116-121.

[6] Greenberg J., Shitovitz B. (1977), "Advantageous monopolies," *Journal of Economic Theory* **16**, 394-402.

[7] Giraud G. (2003), "Strategic market games: an introduction," *Journal of Mathematical Economics* **39**, 355-375.

[8] Kats A. (1974), "Monopolistic trading economies: a case of governmental control," *Public Choice* **20**, 17-32.

[9] Okuno M., Postlewaite A., Roberts J. (1980), "Oligopoly and competition in large markets," *American Economic Review* **70**, 22-31.

[10] Sadanand V. (1988), "Endogenously determined price-setting monopoly in an exchange economy," *Journal of Economic Theory* **46**, 172-178.

[11] Schydlowsky D.M., Siamwalla A. (1966), "Monopoly under general equilibrium: a geometric exercise," *Quarterly Journal of Economics* **80**, 147-153.

[12] Shitovitz B. (1973), "Oligopoly in markets with a continuum of traders," *Econometrica* **41**, 467-501.

[13] Varian H.R. (2014), *Intermediate microeconomics with calculus*, Norton, New York.