Three Models of Noncooperative Oligopoly in Markets with a Continuum of Traders

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Abstract

In this paper, we reconstruct the main developments of the theory of noncooperative oligopoly in general equilibrium, by focusing on the analysis of three prototypical models: the model of Cournot-Walras equilibrium of Codognato and Gabszewicz (1991); the model of Cournot-Nash equilibrium originally proposed by Lloyd S. Shapley and known as window model; the model of Cournot-Walras equilibrium of Busetto et al. (2008). We establish, in a systematic way, the relationship between the three notions of equilibrium proposed in these models and the notion of Walras equilibrium. Then, we investigate the relationships among those three notions of equilibrium.

Journal of Economic Literature Classification Numbers: C72, D51.

1 Introduction

In this paper, we propose a reconstruction of the theory of noncooperative oligopoly in general equilibrium. This theory has been developed along two main lines of research. The first is the Cournot-Walras approach, initiated, in the context of economies with production, by Gabszewicz and Vial (1972), and, in the case of exchange economies, by Codognato and Gabszewicz (1991) (see also Codognato and Gabszewicz (1993), d'Aspremont et

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al. (1997), Gabszewicz and Michel (1997), Shitovitz (1997), Lahmandi-Ayed (2001), Bonnisseau and Florig (2003), Julien and Tricou (1995), Busetto et al. (2008), among others). The second is the noncooperative market game approach, initiated by Shapley and Shubik (1977) (see also Dubey and Shubik (1977), Postlewaite and Schmeidler (1978), Okuno et al. (1980), Mas-Colell (1982), Sahi and Yao (1989), Amir et al. (1990), Peck et al. (1992), Dubey and Shapley (1994), Busetto et al. (2011), among others).

A relevant part of the work elaborated within these two lines of research, and most of the interactions between them, have been concerned with the issue of the strategic foundation of the noncooperative oligopolistic behavior in general equilibrium. As stressed by Okuno et al. (1980), an appropriate model of oligopoly in general equilibrium should give a formal explanation of "[...] either perfectly or imperfectly competitive behavior may emerge endogenously [...], depending on the characteristics of the agent and his place in the economy" (see p. 22).

In this paper, we reconsider this issue by concentrating on three prototypical models proposed in the literature, which are, in our view, the most representative of the two lines of research: the model of Codognato and Gabszewicz (1991), based on a concept of Cournot-Walras equilibrium which we will call, following Gabszewicz and Michel (1997), homogeneous oligopoly equilibrium; the model of Cournot-Nash equilibrium originally proposed by Lloyd S. Shapley and further analyzed by Sahi and Yao (1989) and Busetto et al. (2011); the model of Cournot-Walras equilibrium introduced by Busetto et al. (2008). All these models were originally formalized in a one-stage setting.

We first establish, in a systematic way, the relationship of the three concepts of equilibrium proposed in these models with the notion of Walras equilibrium; to this end, we consider, according to Aumann (1964), limit exchange economies, i.e., markets with an atomless continuum of traders and, according to Shitovitz (1973), mixed exchange economies, i.e., markets with a continuum of traders and atoms. Second, we investigate the relationship among those three concepts of equilibrium.

We reach the conclusion that the three notions of equilibrium are all distinct. In particular, we show that the notion of Cournot-Walras equilibrium introduced by Codognato and Gabszewicz (1991) differs from the notion proposed by Busetto et al. (2008). Moreover, we argue that the Shapley's window model with an atomless continuum of traders and atoms (see Busetto et al. (2011)) is an autonomous description of the one-shot oligopolistic interaction in a general equilibrium framework, since even its

closest Cournot-Walras variant, proposed by Busetto et al. (2008), may generate different equilibria. At the state of the art, it is the only model of noncooperative oligopoly which, according to Okuno et al. (1980), provides an endogenous explanation of the perfectly and imperfectly competitive behavior of agents in a one-stage setting.

As regards the Cournot-Walras approach, instead, the model of Busetto et al. (2008) turns out to be a well-founded representation of the noncooperative oligopolistic interaction in general equilibrium, but only in a two-stage setting. This makes clear a fundamental characteristic of the Cournot-Walras equilibrium concept, namely its two-stage nature, which had remained implicit in all the previous models elaborated within this approach.

The paper is organized as follows. In Section 2, we build the mathematical model of a pure exchange economy where the space of traders is represented by a measure space with atoms and an atomless part. This model allows us to analyze, within a unifying structure, the different models proposed in the literature on noncooperative oligopoly in general equilibrium. In Sections 3, 4, and 5, we introduce, respectively, the concept of homogeneous oligopoly equilibrium of Codognato and Gabszewicz (1991), the concept of Cournot-Nash equilibrium of the Shapley's window model developed by Busetto et al. (2011), and the concept of Cournot-Walras equilibrium of Busetto et al. (2008), and we analyze the relationship of each of them with the notion of Walras equilibrium. In Section 6, we compare the three different concepts of equilibrium.

2 The mathematical model

We consider a pure exchange economy with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space $(T, \mathcal{T}, \mu)$, where T is the set of traders, $\mathcal{T}$ is the σ-algebra of all μ-measurable subsets of T, and μ is a real valued, non-negative, countably additive measure defined on $\mathcal{T}$. We assume that $(T, \mathcal{T}, \mu)$ is finite, i.e., $\mu(T) < \infty$. This implies that the measure space $(T, \mathcal{T}, \mu)$ contains at most countably many atoms. Let T_1 denote the set of atoms and $T_0 = T \setminus T_1$ the atomless part of T. A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for "each" trader in a certain set is to be understood to hold for all such traders except possibly for a null set of

traders. The word "integrable" is to be understood in the sense of Lebesgue.

In the exchange economy, there are l different commodities. A commodity bundle is a point in R_+^l. An assignment (of commodity bundles to traders) is an integrable function $\mathbf{x}\colon T \to R_+^l$. There is a fixed initial assignment $\mathbf{w}$, satisfying the following assumption.

Assumption 1. $\mathbf{w}(t) > 0$, for each $t \in T$, $\int_T \mathbf{w}(t)\, d\mu \gg 0$.

An allocation is an assignment $\mathbf{x}$ for which $\int_T \mathbf{x}(t)\, d\mu = \int_T \mathbf{w}(t)\, d\mu$.

The preferences of each trader $t \in T$ are described by a utility function $u_t : R_+^l \to R$, satisfying the following assumptions.

Assumption 2. $u_t : R_+^l \to R$ is continuous, strongly monotone, and quasi-concave for each $t \in T$.

Let $\mathcal{B}(R_+^l)$ denote the Borel σ-algebra of R_+^l. Moreover, let $\mathcal{T} \bigotimes \mathcal{B}$ denote the σ-algebra generated by the sets $E \times F$, where $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

Assumption 3. $u : T \times R_+^l \to R$, given by $u(t,x) = u_t(x)$, for each $t \in T$ and for each $x \in R_+^l$, is $\mathcal{T} \bigotimes \mathcal{B}$-measurable.

A price vector is a vector $p \in R_+^l$. According to Aumann (1966), we define, for each $p \in R_+^l$, a correspondence $\mathbf{\Delta}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\mathbf{\Delta}_p(t) = \{x \in R_+^l : px \leq p\mathbf{w}(t)\}$, a correspondence $\mathbf{\Gamma}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\mathbf{\Gamma}_p(t) = \{x \in R_+^l : \text{ for all } y \in \mathbf{\Delta}_p(t), u_t(x) \geq u_t(y)\}$, and finally a correspondence $\mathbf{X}_p : T \to \mathcal{P}(R^l)$ such that, for each $t \in T$, $\mathbf{X}_p(t) = \mathbf{\Delta}_p(t) \cap \mathbf{\Gamma}_p(t)$.

A Walras equilibrium is a pair $(p^*, \mathbf{x}^*)$, consisting of a price vector p^* and an allocation $\mathbf{x}^*$, such that, for all $t \in T$, $\mathbf{x}^*(t) \in \mathbf{X}_{p^*}(t)$.

The following proposition, due to Busetto et al. (2011), generalizes a result previously used by Aumann (1966) to prove his existence theorem.

Proposition 1. Under Assumptions 1, 2, and 3, for each $p \in R_{++}^l$, there exists an integrable function $\mathbf{x}(\cdot, p) : T \to R_+^l$ such that, for each $t \in T$, $\mathbf{x}(t, p) \in \mathbf{X}_p(t)$.

3   Homogeneous oligopoly equilibrium

We first consider the notion of Cournot-Walras equilibrium for exchange economies proposed by Codognato and Gabszewicz (1991). The concept of Cournot-Walras equilibrium was originally introduced by Gabszewicz and

Vial (1972) in the framework of an economy with production. These authors were already aware that their notion of equilibrium raised some theoretical difficulties, as it depends on the rule chosen to normalize prices and profit maximization may not be a rational objective of firms. The reformulation of the Cournot-Walras equilibrium for exchange economies proposed by Codognato and Gabszewicz (1991) made it possible to overcome these problems, since it does not depend on price normalization and replaces profit maximization with utility maximization. This concept was generalized by Gabszewicz and Michel (1997) by means of a notion of oligopoly equilibrium for exchange economies (see also d'Aspremont et al. (1997), for another generalization of the concept). More precisely, the Cournot-Walras equilibrium introduced by Codognato and Gabszewicz (1991) corresponds to the case of "homogeneous oligopoly" equilibrium in the framework developed by Gabszewicz and Michel (1997). We adopt here this expression, to distinguish this concept of Cournot-Walras equilibrium from the other proposed by Busetto et al. (2008).

In order to formulate the concept of homogeneous oligopoly equilibrium, we assume that the space of traders is as in Section 2, while the initial assignment of atoms is specified as follows:

$$\mathbf{w}(t) = (\mathbf{w}^1(t), 0, \ldots, 0),$$

for each $t \in T_1$.

Moreover, we need to introduce the following restriction of Assumption 2.

Assumption 2′. $u_t : R_+^l \to R$ is continuous, strongly monotone, and strictly quasi-concave, for each $t \in T$.

Under Assumption 2′, for each $p \in R_{++}^l$, we define the small traders' Walrasian demands as a function $\mathbf{x}^0(\cdot, p) : T_0 \to R_+^l$, such that, for each $t \in T_0$, $\mathbf{x}^0(t, p) = X_p(t)$. It is also possible to show the following proposition.

Proposition 2. Under Assumptions 1, 2′, and 3, the function $\mathbf{x}^0(\cdot, p)$ is integrable, for each $p \in R_{++}^l$.

Proof. By Proposition 1, for each $p \in R_{++}^l$, there exists an integrable function $\mathbf{x}_p : T \to R_+^l$ such that, for each $t \in T$, $\mathbf{x}(t, p) = \mathbf{X}_p(t)$. Then, for each $p \in R_{++}^l$, $\mathbf{x}^0(\cdot, p)$ is integrable as it is a restriction of the integrable function $\mathbf{x}(\cdot, p)$ to T_0. ∎

Consider now the atoms' strategies. A strategy correspondence is a correspondence $\mathbf{Y} : T_1 \to \mathcal{P}(R)$ such that, for each $t \in T_1$, $\mathbf{Y}(t) = \{y \in R :$

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$0 \leq y \leq \mathbf{w}^1(t)\}$. A strategy selection is an integrable function $\mathbf{y} : T_1 \to R$ such that, for each $t \in T_1$, $\mathbf{y}(t) \in \mathbf{Y}(t)$. For each $t \in T_1$, $\mathbf{y}(t)$ represents the amount of commodity 1 that trader t offers in the market. We denote by $\mathbf{y} \setminus y(t)$ a strategy selection obtained by replacing $\mathbf{y}(t)$ in $\mathbf{y}$ with $y \in \mathbf{Y}(t)$. With a slight abuse of notation, $\mathbf{y} \setminus y(t)$ will also represent the value of the strategy selection $\mathbf{y} \setminus y(t)$ at t.

Given a price vector $p \in R^l_{++}$ and a strategy selection $\mathbf{y}$, let $\mathbf{x}^1(\cdot, \mathbf{y}(\cdot), p) : T_1 \to R^l_+$ denote a function such that, for each $t \in T_1$, $\mathbf{x}^{11}(t, \mathbf{y}(t), p) = \mathbf{w}^1(t) - \mathbf{y}(t)$ and $(\mathbf{x}^{12}(t, \mathbf{y}(t), p), \ldots, \mathbf{x}^{1l}(t, \mathbf{y}(t), p))$ is, under Assumption 2′, the unique solution to the problem

$$\max_{x^2,\ldots,x^l} u_t(\mathbf{w}^1(t) - \mathbf{y}(t), x^2, \ldots, x^l) \text{ such that } \sum_{j=2}^{l} p^j x^j = p^1 \mathbf{y}(t).$$

Let $\pi(\mathbf{y})$ denote the correspondence which associates, with each strategy selection $\mathbf{y}$, the set of the price vectors such that

$$\int_{T_0} \mathbf{x}^{01}(t, p) \, d\mu = \int_{T_0} \mathbf{w}^{01}(t) \, d\mu + \int_{T_1} \mathbf{y}(t) \, d\mu,$$

$$\int_{T_0} \mathbf{x}^{0j}(t, p) \, d\mu + \int_{T_1} \mathbf{x}^{1j}(t, \mathbf{y}(t), p) \, d\mu = \int_{T_0} \mathbf{w}^{0j}(t) \, d\mu,$$

$j = 2, \ldots, l$. We assume that, for each $\mathbf{y}$, $\pi(\mathbf{y}) \neq \emptyset$ and $\pi(\mathbf{y}) \subset R^l_{++}$. A price selection $p(\mathbf{y})$ is a function which associates, with each $\mathbf{y}$, a price vector $p \in \pi(\mathbf{y})$.

Given a strategy selection $\mathbf{y}$, by the structure of the traders' measure space, Proposition 2, and the atoms' maximization problem, it is straightforward to show that the function $\mathbf{x}(t)$ such that $\mathbf{x}(t) = \mathbf{x}^0(t, p(\mathbf{y}))$, for each $t \in T_0$, and $\mathbf{x}(t) = \mathbf{x}^1(t, \mathbf{y}(t), p(\mathbf{y}))$, for each $t \in T_1$, is an allocation.

At this stage, we are able to define the concept of homogeneous oligopoly equilibrium.

Definition 1. *A pair* $(\check{\mathbf{y}}, \check{\mathbf{x}})$, *consisting of a strategy selection* $\check{\mathbf{y}}$ *and an allocation* $\check{\mathbf{x}}$ *such that* $\check{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\check{\mathbf{y}}))$, *for each* $t \in T_0$, *and* $\check{\mathbf{x}}(t) = \mathbf{x}^1(t, \check{\mathbf{y}}(t), p(\check{\mathbf{y}}))$, *for each* $t \in T_1$, *is a homogeneous oligopoly equilibrium, with respect to a price selection* $p(\mathbf{y})$, *if* $u_t(\mathbf{x}^1(t, \check{\mathbf{y}}(t), p(\check{\mathbf{y}}))) \geq u_t(\mathbf{x}^1(t, \check{\mathbf{y}} \setminus y(t), p(\check{\mathbf{y}} \setminus y(t))))$, *for each* $y \in \mathbf{Y}(t)$ *and for each* $t \in T_1$.

Let us consider now the relationship between the concepts of homogeneous oligopoly and Walras equilibrium. As is well known, within the

Cournotian tradition (see Cournot (1838)), it has been established that the Cournot equilibrium approaches the competitive equilibrium as the number of oligopolists increases. Codognato and Gabszewicz (1993) confirmed this result. By considering a limit exchange economy à la Aumann, they were able to show that the set of the homogeneous oligopoly equilibrium allocations coincides with the set of the Walras equilibrium allocations.

On the other hand, they provided an example showing that this result no longer holds in an exchange economy à la Shitovitz, where strategic traders are represented as atoms. More precisely, by means of this example they proved that, within their Cournot-Walras structure, it is possible to avoid a counterintuitive result obtained by Shitovitz (1973) in the cooperative context: in his Theorem B, this author proved that the core allocations of a mixed exchange economy are Walrasian when the atoms have the same endowments and preferences (but not necessarily the same measure). Codognato and Gabszewicz (1993) considered an exchange economy with two identical atoms facing an atomless continuum of small traders and showed that, in this economy, there is a homogeneous oligopoly equilibrium allocation which is not Walrasian.

4 Cournot-Nash equilibrium

The model described in the previous section shares, with the whole Cournot-Walras approach, a fundamental problem, stressed, in particular, by Okuno et al. (1980): it does not explain why a certain agent behaves strategically rather than competitively.

Taking inspiration from the cooperative approach to oligopoly introduced by Shitovitz (1973), Okuno et al. (1980) proposed a foundation of agents' behavior based on the Cournot-Nash equilibria of a model of simultaneous, noncooperative exchange between large traders, represented as atoms, and small traders, represented by an atomless sector. Their model belongs to the line of research initiated by Shapley and Shubik (1977). In their framework, all agents interact strategically, but only part of them turns out to be price taker while the others have influence on prices, depending on their characteristics and their weight in the economy. Okuno et al. (1980) were also the first who showed that the unsatisfying result obtained by Shitovitz (1973) with his Theorem B could be avoided in the noncooperative context: they gave both an example and a proposition showing that, in their Cournot-Nash equilibrium model, the small traders always have a negligible

influence on prices, while the large traders keep their strategic power even when their behavior turns out to be Walrasian in the cooperative framework considered by Shitovitz (1973). Nevertheless, the model they used incorporates very special hypotheses, since it considers only two commodities that no trader can simultaneously buy and sell.

In this section, we show a similar result within the more general model of simultaneous, noncooperative exchange originally proposed by Lloyd S. Shapley and subsequently analyzed by Sahi and Yao (1989) in the case of exchange economies with a finite number of traders, Codognato and Ghosal (2000) in the case of limit exchange economies, and Busetto et al. (2011) in the case of mixed exchange economies à la Shitovitz. We formalize the model following Busetto et al. (2011).

We assume that the space of traders is as in Section 2. Moreover, we introduce the following further assumption (see also Sahi and Yao (1989)).

Assumption 4. *There are at least two traders in T_1 for whom $\mathbf{w}(t) \gg 0$; u_t is continuously differentiable in R_{++}^l; $\{x \in R_+^l : u_t(x) = u_t(\mathbf{w}(t))\} \subset R_{++}^l$.*

Consider now the traders' strategies. Let $b \in R^{l^2}$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{ll-1}, b_{ll})$. A strategy correspondence is a correspondence $\mathbf{B} : T \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T$, $\mathbf{B}(t) = \{b \in R^{l^2} : b_{ij} \geq 0, i, j = 1, \ldots, l; \sum_{j=1}^l b_{ij} \leq \mathbf{w}^i(t), i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{b} : T \to R^{l^2}$, such that, for each $t \in T$, $\mathbf{b}(t) \in \mathbf{B}(t)$. For each $t \in T$, $\mathbf{b}_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. Given a strategy selection $\mathbf{b}$, we define the aggregate matrix $\bar{\mathbf{B}} = (\int_T \mathbf{b}_{ij}(t) \, d\mu)$. Moreover, we denote by $\mathbf{b} \setminus b(t)$ a strategy selection obtained by replacing $\mathbf{b}(t)$ in $\mathbf{b}$ with $b \in \mathbf{B}(t)$. With a slight abuse of notation, $\mathbf{b} \setminus b(t)$ will also represent the value of the strategy selection $\mathbf{b} \setminus b(t)$ at t.

Then, we introduce two further definitions (see Sahi and Yao (1989)).

Definition 2. *A nonnegative square matrix A is said to be irreducible if, for every pair (i, j), with $i \neq j$, there is a positive integer $k = k(i, j)$ such that $a_{ij}^{(k)} > 0$, where $a_{ij}^{(k)}$ denotes the ij-th entry of the k-th power A^k of A.*

Definition 3. *Given a strategy selection $\mathbf{b}$, a price vector p is market clearing if*

$$p \in R_{++}^l, \sum_{i=1}^l p^i \bar{\mathbf{b}}_{ij} = p^j (\sum_{i=1}^l \bar{\mathbf{b}}_{ji}), j = 1, \ldots, l. \tag{1}$$

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By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector p satisfying (1) if and only if $\bar{\mathbf{B}}$ is irreducible. Then, we denote by $p : R_+^{l^2} \to R_+^l$ a function such that, for each strategy selection $\mathbf{b}$, $p(\mathbf{b})$ is the unique, up to a scalar multiple, price vector satisfying (1), if $\bar{\mathbf{B}}$ is irreducible, and $p(\mathbf{b}) = 0$, otherwise.

Given a strategy selection $\mathbf{b}$ and a price vector p, consider the assignment determined as follows:

$$
\begin{aligned}
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{b}_{ji}(t) + \sum_{i=1}^{l} \mathbf{b}_{ij}(t) \frac{p^i}{p^j}, \text{ if } p \in R_{++}^l, \\
\mathbf{x}^j(t, \mathbf{b}(t), p) &= \mathbf{w}^j(t), \text{ otherwise,}
\end{aligned}
$$

$j = 1, \ldots, l$, for each $t \in T$.

According to this rule, given a strategy selection $\mathbf{b}$ and the function $p(\cdot)$, the traders' final holdings are determined as follows:

$$
\mathbf{x}(t) = \mathbf{x}(t, \mathbf{b}(t), p(\mathbf{b})),
$$

for each $t \in T$. It is straightforward to show that the assignment corresponding to the final holdings is an allocation.

This reformulation of the Shapley's window model allows us to define the following concept of Cournot-Nash equilibrium for mixed exchange economies.

Definition 4. *A strategy selection $\hat{\mathbf{b}}$ such that $\hat{\bar{\mathbf{B}}}$ is irreducible is a Cournot-Nash equilibrium if*

$$
u_t(\mathbf{x}(t, \hat{\mathbf{b}}(t), p(\hat{\mathbf{b}}))) \geq u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\hat{\mathbf{b}} \setminus b(t)))),
$$

for all $b \in \mathbf{B}(t)$ and for each $t \in T$.

Busetto et al. (2011) proved that, under Assumptions 1, 2, 3, and 4, there exists a Cournot-Nash equilibrium $\hat{\mathbf{b}}$. Moreover, Codognato and Ghosal (2000) showed that, in limit exchange economies, the set of the Cournot-Nash equilibrium allocations of the Shapley's model and the set of the Walras equilibrium allocations coincide. Here, we deal with the question whether, according to Okuno et al. (1980), this equivalence no longer holds under the assumptions of Theorem B in Shitovitz (1973): we provide a proposition and an example showing that, under those assumptions, the small traders always have a Walrasian price-taking behavior whereas the large traders have market power even in those circumstances where the core outcome is competitive.

Proposition 3. *Assumptions 1, 2, 3, and 4, for each strategy selection* $\mathbf{b}$ *such that* $\bar{\mathbf{B}}$ *is irreducible and for each* $t \in T_0$, *(i)* $p(\mathbf{b}) = p(\mathbf{b} \setminus b(t))$, *for all* $b \in \mathbf{B}(t)$; *(ii)* $\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\mathbf{b} \setminus b(t))) \in \mathbf{X}_{p(\mathbf{b})}(t)$, *for all* $b \in \arg\max\{u_t(\mathbf{x}(t, \hat{\mathbf{b}} \setminus b(t), p(\mathbf{b} \setminus b(t)))) : b \in \mathbf{B}(t)\}$.

Proof. (i) It is an immediate consequence of Definition 3. (ii) It can be proved by the same argument used in the proof of part (i) of Theorem 2 in Codognato and Ghosal (2000). ∎

More precisely, part (i) of Proposition 3 establishes that each small trader is unable to influence prices and part (ii) that all the best replies of each small trader attains a point in his Walras demand correspondence.

Example 1. *Consider the following specification of an exchange economy satisfying Assumptions 1, 2, 3, and 4, where* $l = 2$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(2) = \mathbf{w}(3)$, $u_2(x) = u_3(x)$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = (x^1)^\alpha (x^2)^{1-\alpha}$, $0 < \alpha < 1$, *for each* $t \in T_0$. *Then, if* $\hat{\mathbf{b}}$ *is a Cournot-Nash equilibrium, the pair* $(\hat{p}, \hat{\mathbf{x}})$ *such that* $\hat{p} = p(\hat{\mathbf{b}})$ *and* $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, *for each* $t \in T$, *is not a Walras equilibrium.*

Proof. Suppose that $\hat{\mathbf{b}}$ is a Cournot-Nash equilibrium and that the pair $(\hat{p}, \hat{\mathbf{x}})$ such that $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for all $t \in T$, is a Walras equilibrium. Clearly, $\hat{\mathbf{b}}_{21}(t) = \alpha$, for each $t \in T_0$. At a Nash equilibrium, the marginal price of each atom must be equal to his marginal rate of substitution between commodity 1 and commodity 2 (see Okuno et al. (1980)). This, in turn, at a Walras equilibrium, must be equal to the relative price of commodity 1 in terms of commodity 2. Consequently, we must have

$$\frac{dx_2}{dx_1} = -\hat{p}^2 \frac{\hat{\mathbf{b}}_{12}(t)}{\hat{\mathbf{b}}_{21}(t) + \alpha} = -\hat{p},$$

for each $t \in T_1$. But then, we must also have

$$\frac{\hat{\mathbf{b}}_{21}(2) + \alpha}{\hat{\mathbf{b}}_{12}(2)} = \frac{\hat{\mathbf{b}}_{21}(2) + \hat{\mathbf{b}}_{21}(3) + \alpha}{\hat{\mathbf{b}}_{12}(2) + \hat{\mathbf{b}}_{12}(3)} = \frac{\hat{\mathbf{b}}_{21}(3) + \alpha}{\hat{\mathbf{b}}_{12}(3)}. \tag{2}$$

The last equality in (2) holds if and only if $\hat{\mathbf{b}}_{21}(2) = k(\hat{\mathbf{b}}_{21}(3) + \alpha)$ and $\hat{\mathbf{b}}_{12}(2) = k\hat{\mathbf{b}}_{12}(3)$, with $k > 0$. But then, the first and the last members in (2) cannot be equal because

$$\frac{k(\hat{\mathbf{b}}_{21}(3) + \alpha) + \alpha}{k\hat{\mathbf{b}}_{12}(3)} \neq \frac{\hat{\mathbf{b}}_{21}(3) + \alpha}{\hat{\mathbf{b}}_{12}(3)}.$$

This implies that the pair $(\hat{p}, \hat{\mathbf{x}})$ such that $\hat{p} = p(\hat{\mathbf{b}})$ and $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}(t), \hat{p})$, for each $t \in T$, cannot be a Walras equilibrium. ∎

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Notice that our example provides a result stronger than the proposition proved by Okuno et al. (1980), since it requires that atoms have not only the same endowments and preferences but also the same measure.

Proposition 3 and Example 1 clarify that the mixed version of the Shapley's model introduced in this section is a well-founded model of oligopoly in general equilibrium as it gives an endogenous explanation of strategic and competitive behavior. Therefore, it is immune from the criticism by Okuno et al. (1980).

To conclude this section, let us mention a recent result by Busetto et al. (2012), on the convergence of the Cournot-Nash equilibrium of the Shapley's model to the Walras equilibrium. In the original spirit of Cournot (1838), these authors replicate only the set of atoms while keeping unchanged the atomless part of the economy. They show that any convergent sequence of Cournot-Nash equilibrium allocations of the strategic market game à la Shapley associated with the partially replicated exchange economies approximates a Walras equilibrium allocation of the original exchange economy.

5   Cournot-Walras equilibrium

In this section, we describe the concept of Cournot-Walras equilibrium introduced by Busetto et al. (2008), which proposes some emendations to the notion of equilibrium introduced by Codognato and Gabszewicz (1991). In particular, within the model proposed by these authors, oligopolists are characterized by a "twofold behavior," since they act à la Cournot in making their supply decisions and à la Walras in exchanging commodities. As we shall see below, in the model of Busetto et al. (2008), oligopolists always behave à la Cournot. This model can be viewed as a respecification à la Cournot-Walras of the mixed version of the Shapley's model presented in Section 4.

In order to formulate the concept of Cournot-Walras equilibrium, we assume again that the space of traders is as in Section 2. Moreover, we need to introduce the following restriction of Assumption 1.

Assumption 1'. $\mathbf{w}(t) > 0$, for each $t \in T$, $\int_{T_0} \mathbf{w}(t)\, d\mu \gg 0$.

As regards the atomless sector, like in Section 3 we define, under Assumption 2', the Walrasian demands as a function $\mathbf{x}^0(\cdot, p) : T_0 \to R_+^l$ such that $\mathbf{x}^0(t, p) = \mathbf{X}_p(t)$, for each $t \in T_0$.

Consider now the atoms' strategies. Let $e \in R^{l^2}$ be a vector such that $e = (e_{11}, e_{12}, \ldots, e_{ll-1}, e_{ll})$. A strategy correspondence is a correspondence

$\mathbf{E} : T_1 \rightarrow \mathcal{P}(R^{l^2})$ such that, for each $t \in T_1$, $\mathbf{E}(t) = \{e \in R^{l^2} : e_{ij} \geq 0,\ i, j = 1, \ldots, l;\ \sum_{j=1}^{l} e_{ij} \leq \mathbf{w}^i(t),\ i = 1, \ldots, l\}$. A strategy selection is an integrable function $\mathbf{e} : T_1 \rightarrow R^{l^2}$ such that, for each $t \in T_1$, $\mathbf{e}(t) \in \mathbf{E}(t)$. For each $t \in T_1$, $\mathbf{e}_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity i that trader t offers in exchange for commodity j. We denote by $\mathbf{e} \setminus e(t)$ a strategy selection obtained by replacing $\mathbf{e}(t)$ in $\mathbf{e}$ with $e \in \mathbf{E}(t)$. With a slight abuse of notation, $\mathbf{e} \setminus e(t)$ will also represent the value of the strategy selection $\mathbf{e} \setminus e(t)$ at t.

Finally, we denote by $\pi(\mathbf{e})$ the correspondence which associates, with each $\mathbf{e}$, the set of the price vectors such that

$$\int_{T_0} \mathbf{x}^{0j}(t, p)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}_{ij}(t)\, d\mu \frac{p^i}{p^j} = \int_{T_0} \mathbf{w}^j(t)\, d\mu + \sum_{i=1}^{l} \int_{T_1} \mathbf{e}_{ji}(t)\, d\mu, \quad (3)$$

$j = 1, \ldots, l$.

Assumption 5. *For each* $\mathbf{e}$, $\pi(\mathbf{e}) \neq \emptyset$ *and* $\pi(\mathbf{e}) \subset R^l_{++}$.

A price selection $p(\mathbf{e})$ is a function which associates, with each $\mathbf{e}$, a price vector $p \in \pi(\mathbf{e})$ and is such that $p(\mathbf{e}') = p(\mathbf{e}'')$ if $\int_{T_1} \mathbf{e}'(t)\, d\mu = \int_{T_1} \mathbf{e}''(t)\, d\mu$. For each strategy selection $\mathbf{e}$, let $\mathbf{x}^1(\cdot, \mathbf{e}(\cdot), p(\mathbf{e})) : T_1 \rightarrow R^l_+$ denote a function such that

$$\mathbf{x}^{1j}(t, \mathbf{e}(t), p(\mathbf{e})) = \mathbf{w}^j(t) - \sum_{i=1}^{l} \mathbf{e}_{ji}(t) + \sum_{i=1}^{l} \mathbf{e}_{ij}(t) \frac{p^i(\mathbf{e})}{p^j(\mathbf{e})}, \qquad (4)$$

$j = 1, \ldots, l$, for each $t \in T_1$. Given a strategy selection $\mathbf{e}$, taking into account the structure of the traders' measure space, Proposition 2, and Equation (3), it is straightforward to show that the function $\mathbf{x}(t)$ such that $\mathbf{x}(t) = \mathbf{x}^0(t, p(\mathbf{e}))$, for each $t \in T_0$, and $\mathbf{x}(t) = \mathbf{x}^1(t, \mathbf{e}(t), p(\mathbf{e}))$, for each $t \in T_1$, is an allocation.

At this stage, we are able to define the concept of Cournot-Walras equilibrium.

Definition 5. *A pair* $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, *consisting of a strategy selection* $\tilde{\mathbf{e}}$ *and an allocation* $\tilde{\mathbf{x}}$ *such that* $\tilde{\mathbf{x}}(t) = \mathbf{x}^0(t, p(\tilde{\mathbf{e}}))$, *for each* $t \in T_0$, *and* $\tilde{\mathbf{x}}(t) = \mathbf{x}^1(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))$, *for each* $t \in T_1$, *is a Cournot-Walras equilibrium, with respect to a price selection* $p(\mathbf{e})$, *if*

$$u_t(\mathbf{x}^1(t, \tilde{\mathbf{e}}(t), p(\tilde{\mathbf{e}}))) \geq u_t(\mathbf{x}^1(t, \tilde{\mathbf{e}} \setminus e(t), p(\tilde{\mathbf{e}} \setminus e(t)))),$$

for each $e \in \mathbf{E}(t)$ and for each $t \in T_1$.

Let us investigate the relationship between the concepts of Cournot-Walras and Walras equilibrium. Here, we show, for the Cournot-Walras equilibrium, a result similar to those obtained, in limit exchange economies, by Codognato and Gabszewicz (1993) for the homogeneous oligopoly equilibrium, and by Codognato and Ghosal (2000) for the Cournot-Nash equilibrium. More precisely, we assume that T_1 is nonempty and atomless.

The following proposition establishes that, in this framework, the set of the Cournot-Walras equilibrium allocations coincides with the set of the Walras equilibrium allocations.

Proposition 4. *Under Assumptions 1′, 2′, 3, and 5, (i) if $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ is a Cournot-Walras equilibrium with respect to a price selection $p(\mathbf{e})$, there is a price vector $\tilde{p}$ such that $(\tilde{p}, \tilde{\mathbf{x}})$ is a Walras equilibrium; (ii) if $(p^*, \mathbf{x}^*)$ is a Walras equilibrium, there is a strategy selection $\mathbf{e}^*$ such that $(\mathbf{e}^*, \mathbf{x}^*)$ is a Cournot-Walras equilibrium with respect to a price selection $p(\mathbf{e})$.*

Proof. (i) Let $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ be a Cournot-Walras equilibrium with respect to the price selection $p(\mathbf{e})$. First, it is straightforward to show that, for each $t \in T_1$, $\tilde{p}\tilde{\mathbf{x}}(t) = \tilde{p}\mathbf{w}(t)$, where $\tilde{p} = p(\tilde{\mathbf{e}})$. Let us now show that, for each $t \in T_1$, $\mathbf{X}_{\tilde{p}}(t)$. Suppose that this is not the case for a trader $\tau \in T_1$. Then, by Assumption 2′, there is a bundle $\bar{x} \in \{x \in R_+^l : \tilde{p}x = \tilde{p}\mathbf{w}(\tau)\}$ such that $u_\tau(\bar{x}) > u_\tau(\tilde{\mathbf{x}}(\tau))$. By Lemma 5 in Codognato and Ghosal (2000), there exist $\lambda^j \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \lambda^j = 1$, such that

$$\bar{x}^j = \lambda^j \frac{\sum_{i=1}^l \tilde{p}^i \mathbf{w}^i(\tau)}{\tilde{p}^j}, \, j = 1, \ldots, l.$$

Let $\bar{e}_{ij}(\tau) = \mathbf{w}^i(\tau)\lambda^j$, $i, j = 1, \ldots, l$. Substituting in Equation (4) and taking into account the fact that, by Equation (3), $p(\tilde{\mathbf{e}}) = p(\tilde{\mathbf{e}} \setminus \bar{e}(t)) = \tilde{p}$, it is easy to verify that

$$\mathbf{x}^{1j}(\tau, \tilde{\mathbf{e}} \setminus \bar{e}(\tau), p(\tilde{\mathbf{e}} \setminus \bar{e}(\tau))) = \mathbf{w}^j(\tau) - \sum_{i=1}^l \mathbf{w}^j(\tau)\lambda^i + \sum_{i=1}^l \mathbf{w}^i(\tau)\lambda^j \frac{\tilde{p}^i}{\tilde{p}^j} = \bar{x}^j,$$

$j = 1, \ldots, l$. But then, we have

$$u_\tau(\mathbf{x}^1(\tau, \tilde{\mathbf{e}} \setminus \bar{e}(\tau), p(\tilde{\mathbf{e}} \setminus \bar{e}(\tau)))) = u_\tau(\bar{x}) > u_\tau(\tilde{\mathbf{x}}(\tau)) = u_\tau(\mathbf{x}^1(\tau, \tilde{\mathbf{e}}(\tau), p(\tilde{\mathbf{e}}))),$$

which contradicts the fact that the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$ is a Cournot-Walras equilibrium. (ii) Let $(p^*, \mathbf{x}^*)$ be a Walras equilibrium. First, notice that, by

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Assumption 2', $p^* \in R^l_{++}$ and $p^* \mathbf{x}^*(t) = p^* \mathbf{w}(t)$, for each $t \in T$. But then, by Lemma 5 in Codognato and Ghosal (2000), for each $t \in T_1$, there exist $\lambda^{*j}(t) \geq 0$, $j = 1, \ldots, l$, $\sum_{j=1}^l \lambda^{*j}(t) = 1$, such that

$$\mathbf{x}^{*j}(t) = \lambda^{*j}(t) \frac{\sum_{i=1}^l p^{*i} \mathbf{w}^i(t)}{p^{*j}}, \, j = 1, \ldots, l.$$

Define now a function $\boldsymbol{\lambda}^* : T_1 \to R^l_+$ such that $\boldsymbol{\lambda}^{*j}(t) = \lambda^{*j}(t)$, $j = 1, \ldots, l$, for each $t \in T_1$ and a function $\mathbf{e}^* : T_1 :\to R^{l^2}_+$ such that $\mathbf{e}^*_{ij}(t) = \mathbf{w}^i(t) \boldsymbol{\lambda}^{*j}(t)$, $i, j = 1, \ldots, l$, for each $t \in T_1$. It is straightforward to show that the function $\mathbf{e}^*$ is integrable. Moreover, by using Equation (4), it is easy to verify that

$$\mathbf{x}^{*j}(t) = \mathbf{w}^j(t) - \sum_{i=1}^l \mathbf{e}^*_{ji}(t) + \sum_{i=1}^l \mathbf{e}^*_{ij}(t) \frac{p^{*i}}{p^{*j}},$$

$j = 1, \ldots, l$, for each $t \in T_1$. As $\mathbf{x}^*$ is an allocation, it follows that

$$\int_{T_0} \mathbf{x}^{*j}(t) \, d\mu + \int_{T_1} \mathbf{w}^j(t) \, d\mu - \sum_{i=1}^l \int_{T_1} \mathbf{e}^*_{ji}(t) \, d\mu + \sum_{i=1}^l \int_{T_1} \mathbf{e}^*_{ij}(t) \, d\mu \frac{p^{*i}}{p^{*j}} =$$

$\int_T \mathbf{w}^j(t) \, d\mu$,
$j = 1, \ldots, l$. This, in turn, implies that

$$\int_{T_0} \mathbf{x}^{*j}(t) \, d\mu + \sum_{i=1}^l \int_{T_1} \mathbf{e}^*_{ij}(t) \, d\mu \frac{p^{*i}}{p^{*j}} = \int_{T_0} \mathbf{w}^j(t) \, d\mu + \sum_{i=1}^l \int_{T_1} \mathbf{e}^*_{ji}(t) \, d\mu,$$

$j = 1, \ldots, l$. But then, by Assumption 5, there is a price selection $p(\mathbf{e})$ such that $p^* = p(\mathbf{e}^*)$ and, consequently, $\mathbf{x}^*(t) = \mathbf{x}^0(t, p(\mathbf{e}^*))$, for each $t \in T_0$, and $\mathbf{x}^*(t) = \mathbf{x}^1(t, \mathbf{e}^*(t), p(\mathbf{e}^*))$, for each $t \in T_1$. It remains to show that no trader $t \in T_1$ has an advantageous deviation from $\mathbf{e}^*$. Suppose, on the contrary, that there exists a trader $\tau \in T_1$ and a strategy $\bar{e} \in \mathbf{E}(\tau)$ such that

$$u_\tau(\mathbf{x}^1(\tau, \mathbf{e}^* \setminus \bar{e}(\tau), p(\mathbf{e}^* \setminus \bar{e}(\tau)))) > u_\tau(\mathbf{x}^1(\tau, \mathbf{e}^*(\tau), p(\mathbf{e}^*))).$$

By Equation (3), we have $p(\mathbf{e}^* \setminus \bar{e}(\tau)) = p(\mathbf{e}^*) = p^*$. Moreover, it is easy to show that $p^* \mathbf{x}^1(\tau, \mathbf{e}^* \setminus \bar{e}(\tau), p(\mathbf{e}^* \setminus \bar{e}(\tau)) = p^* \mathbf{w}(\tau)$. But then, the pair $(p^*, \mathbf{x}^*)$ is not a Walras equilibrium, which generates a contradiction. ∎

Proposition 4 has the following corollary assuring the existence of a Cournot-Walras equilibrium in limit exchange economies.

Corollary. *Under Assumptions 1′, 2′, 3, and 5, a Cournot-Walras equilibrium exists.*

Proof. From Aumann (1966), we know that, under Assumptions 1′, 2′, and 3, a Walras equilibrium exists. But then, by part (ii) of Proposition 4, this implies that a Cournot-Walras equilibrium exists. ∎

The question whether the equivalence between the concepts of Cournot-Walras and Walras equilibrium still holds when the strategic traders are represented as atoms was dealt with by Busetto et al. (2008). They analyzed an exchange economy with two identical atoms facing an atomless continuum of traders and gave an example showing that, in this economy, there is a Cournot-Walras equilibrium allocation which is not Walrasian. We repropose here their example, which we will refer to also in the next section.

Example 2. *Consider the following specification of an exchange economy satisfying Assumptions 1′, 2′, 3, and 5, where $l = 2$, $T_1 = \{2,3\}$, $T_0 = [0,1]$, $\mathbf{w}(t) = (1,0)$, $u_t(x) = lnx^1 + lnx^2$, for each $t \in T_1$, $\mathbf{w}(t) = (1,0)$, $u_t(x) = lnx^1 + lnx^2$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0,1)$, $u_t(x) = lnx^1 + lnx^2$, for each $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot-Walras equilibrium allocation which does not correspond to any Walras equilibrium.*

Proof. The only symmetric Cournot-Walras equilibrium is the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{1+\sqrt{13}}{12}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{11+\sqrt{13}}{12}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{1+\sqrt{13}}{20+8\sqrt{13}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{2}$, $\tilde{\mathbf{x}}^2(t) = \frac{3}{10+4\sqrt{13}}$, for all $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{5+2\sqrt{13}}{6}$, $\tilde{\mathbf{x}}^2(t) = \frac{1}{2}$, for each $t \in [\frac{1}{2}, 1]$. On the other hand, the only Walras equilibrium of the economy considered is the pair $(\mathbf{x}^*, p^*)$, where $\mathbf{x}^{*1}(2) = \mathbf{x}^{*1}(3) = \frac{1}{2}$, $\mathbf{x}^{*2}(2) = \mathbf{x}^{*2}(3) = \frac{1}{10}$, $\mathbf{x}^{*1}(t) = \frac{1}{2}$, $\mathbf{x}^{*2}(t) = \frac{1}{10}$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{x}^{*1}(t) = \frac{5}{2}$, $\mathbf{x}^{*2}(t) = \frac{1}{2}$, for each $t \in [\frac{1}{2}, 1]$, $p^* = \frac{1}{5}$. ∎

Therefore, the counterintuitive result established by Shitovitz (1973) with his Theorem B can be avoided also in the framework of Busetto et al. (2008). In the next section, where we compare the different notions of equilibrium introduced above, we will see how the problem of providing an endogenous explanation of the strategic and competitive behavior within the Cournot-Walras approach has been addressed by using the Cournot-Nash equilibrium of the Shapley's model.

6 Homogeneous oligopoly, Cournot-Nash, and Cournot-Walras equilibrium

In this section, we analyze, in a systematic way, the relationships among the three concepts of equilibrium presented above. We first show that the homogeneous oligopoly equilibrium concept proposed by Codognato and Gabszewicz (1991) and the Cournot-Walras equilibrium concept introduced by Busetto et al. (2008) differ. To this end, we provide the following example showing that there is a Cournot-Walras equilibrium allocation which does not correspond to any homogeneous oligopoly equilibrium.

Example 3. *Consider the following specification of an exchange economy satisfying Assumptions 1$'$, 2$'$, 3, and 5, where $l = 3$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0, 0)$, $u_t(x) = 2x^1 + lnx^2 + lnx^3$, for each $t \in T_1$, $\mathbf{w}(t) = (1, 0, 0)$, $u_t(x) = lnx^1 + lnx^2 + lnx^3$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1, 1)$, $u_t(x) = x^1 + \frac{1}{2}lnx^2 + lnx^3$, for each $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot-Walras equilibrium allocation which does not correspond to any homogeneous oligopoly equilibrium.*

Proof. There is a unique Cournot-Walras equilibrium represented by the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{1+\sqrt{241}}{48}$, $\tilde{\mathbf{e}}_{13}(2) = \tilde{\mathbf{e}}_{13}(3) = \frac{-1+\sqrt{97}}{24}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{49-\sqrt{241}-2\sqrt{97}}{48}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{1+\sqrt{241}}{44+4\sqrt{241}}$, $\tilde{\mathbf{x}}^3(2) = \tilde{\mathbf{x}}^3(3) = \frac{-1+\sqrt{97}}{28+4\sqrt{97}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{3}$, $\tilde{\mathbf{x}}^2(t) = \frac{4}{11+\sqrt{241}}$, $\tilde{\mathbf{x}}^3(t) = \frac{2}{7+\sqrt{97}}$, for each $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{9+\sqrt{97}+\sqrt{241}}{12}$, $\tilde{\mathbf{x}}^2(t) = \frac{6}{11+\sqrt{241}}$, $\tilde{\mathbf{x}}^3(t) = \frac{6}{7+\sqrt{97}}$, for each $t \in [\frac{1}{2}, 1]$. On the other hand, there is no interior symmetric homogeneous oligopoly equilibrium for the economy considered. ∎

Codognato (1995) compared the mixed version of the model in Codognato and Gabszewicz (1991) with the mixed version of the Shapley's model. The point was the following: if the set of the equilibrium allocations of the model of homogeneous oligopoly equilibrium - where strategic and competitive behavior is assumed a priori - had coincided with the set of the equilibrium allocations of the Shapley's model - where strategic and competitive behavior is generated endogenously - then the notion of homogeneous oligopoly equilibrium could have been re-interpreted as the outcome of a game in which all agents behave strategically but those belonging to the atomless sector turn out to act competitively whereas the atoms turn out to have market power. Therefore, this equilibrium concept would have been immune from the criticism by Okuno et al. (1980).

Nonetheless, Codognato (1995) provided an example showing that the set of the homogeneous oligopoly equilibrium allocations does not coincide with the set of the Cournot-Nash equilibrium allocations. There were two possible explanations of this result. The first is that the homogeneous oligopoly equilibrium, like the other Cournot-Walras equilibrium concepts, has an intrinsic two-stage nature, which cannot be reconciled with the one-stage Cournot-Nash equilibrium of the Shapley's model. The second is that, in the model of Codognato and Gabszewicz (1991), the oligopolists have the twofold behavior mentioned above - as they act à la Cournot in making their supply decisions and à la Walras in exchanging commodities - whereas, in the mixed version of the Shapley's model, they always behave à la Cournot.

The relationship between the concepts of Cournot-Walras and Cournot-Nash equilibrium was analyzed by Busetto et al. (2008). They noticed that the allocation corresponding to a Cournot-Walras equilibrium in Example 2 also corresponds to a Cournot-Nash equilibrium as in Definition 4. Consequently, the Cournot-Nash equilibrium can be viewed as a situation in which all traders behave strategically but those belonging to the atomless sector have a negligible influence on prices. In this paper, by means of Example 1 and Proposition 3, we have provided a more general proof of this fact. Thus, the strategic behavior of the atomless sector can be interpreted as competitive behavior.

On the other hand, at a Cournot-Walras equilibrium as in Definition 5, the atomless sector is supposed to behave competitively while the atoms have strategic power. This led Busetto et al. (2008) to conjecture that the set of the Cournot-Walras equilibrium allocations coincides with the set of the Cournot-Nash equilibrium allocations. They showed that this is false by means of the following example.

Example 4. *Consider the following specification of an exchange economy satisfying Assumptions 1$'$, 2$'$, 3, and 4, where $l = 2$, $T_1 = \{2, 3\}$, $T_0 = [0, 1]$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = \ln x^1 + \ln x^2$, for each $t \in T_1$, $\mathbf{w}(t) = (1, 0)$, $u_t(x) = \ln x^1 + \ln x^2$, for each $t \in [0, \frac{1}{2}]$, $\mathbf{w}(t) = (0, 1)$, $u_t(x) = x^1 + \ln x^2$, for each $t \in [\frac{1}{2}, 1]$. For this economy, there is a Cournot-Walras equilibrium allocation which does not correspond to any Cournot-Nash equilibrium.*

Proof. The only symmetric Cournot-Walras equilibrium of the economy considered is the pair $(\tilde{\mathbf{e}}, \tilde{\mathbf{x}})$, where $\tilde{\mathbf{e}}_{12}(2) = \tilde{\mathbf{e}}_{12}(3) = \frac{-1+\sqrt{37}}{12}$, $\tilde{\mathbf{x}}^1(2) = \tilde{\mathbf{x}}^1(3) = \frac{11-\sqrt{37}}{12}$, $\tilde{\mathbf{x}}^2(2) = \tilde{\mathbf{x}}^2(3) = \frac{-1+\sqrt{37}}{14+4\sqrt{37}}$, $\tilde{\mathbf{x}}^1(t) = \frac{1}{2}$, $\tilde{\mathbf{x}}^2(t) = \frac{3}{7+2\sqrt{37}}$, for each $t \in [0, \frac{1}{2}]$, $\tilde{\mathbf{x}}^1(t) = \frac{1+2\sqrt{37}}{6}$, $\tilde{\mathbf{x}}^2(t) = \frac{6}{7+2\sqrt{37}}$, for each $t \in [\frac{1}{2}, 1]$.

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On the other hand, the only symmetric Cournot-Nash equilibrium is the strategy selection $\hat{\mathbf{b}}_{12}(2) = \hat{\mathbf{b}}_{12}(3) = \frac{1+\sqrt{13}}{12}$, $\hat{\mathbf{b}}_{12}(t) = \frac{1}{2}$, for each $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{b}}_{21}(t) = \frac{5+2\sqrt{13}}{11+2\sqrt{13}}$ for each $t \in [\frac{1}{2}, 1]$, which generates the allocation $\hat{\mathbf{x}}^1(2) = \hat{\mathbf{x}}^1(3) = \frac{11+\sqrt{13}}{12}$, $\hat{\mathbf{x}}^2(2) = \hat{\mathbf{x}}^2(3) = \frac{1+\sqrt{13}}{22+4\sqrt{13}}$, $\hat{\mathbf{x}}^1(t) = \frac{1}{2}$, $\hat{\mathbf{x}}^2(t) = \frac{3}{11+2\sqrt{13}}$, for each $t \in [0, \frac{1}{2}]$, $\hat{\mathbf{x}}^1(t) = \frac{5+2\sqrt{13}}{6}$, $\hat{\mathbf{x}}^2(t) = \frac{6}{11+2\sqrt{13}}$, for each $t \in [\frac{1}{2}, 1]$, where $\hat{\mathbf{x}}(t) = \mathbf{x}(t, \hat{\mathbf{b}}, p(\hat{\mathbf{b}}))$, for each $t \in T$. ∎

This confirms, in a more general framework, the nonequivalence result obtained by Codognato (1995). In this regard, it is worth noticing that, in both models compared in Example 4, large traders always behave à la Cournot. Therefore, this example removes one of the possible explanations of the nonequivalence proved by Codognato (1995), namely the twofold behavior of the oligopolists assumed in the model of homogeneous oligopoly equilibrium. This suggested to Busetto et al. (2008) that the general cause of the nonequivalence between the concepts of Cournot-Walras and Cournot-Nash equilibrium had to be the two-stage implicit nature of the Cournot-Walras equilibrium concept, which cannot be reconciled with the one-stage Cournot-Nash equilibrium of the Shapley's model. For this reason, they introduced a reformulation of the Shapley's model as a two-stage game, where the atoms move in the first stage and the atomless sector moves in the second stage, and showed that the set of the Cournot-Walras equilibrium allocations coincides with a specific set of subgame perfect equilibrium allocations of this two-stage game. Therefore, they provided a strategic foundation of the Cournot-Walras approach in a two-stage setting.

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