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Integer Programming on Domains Containing Inseparable Ordered Pairs

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Abstract

Using the integer programming approach introduced by Sethuraman, Teo, and Vohra (2003), we extend the analysis of the preference domains containing an inseparable ordered pair, initiated by Kalai and Ritz (1978). We show that these domains admit not only Arrovian social welfare functions "without ties," but also Arrovian social welfare functions "with ties," since they satisfy the strictly decomposability condition introduced by Busetto, Codognato, and Tonin (2012). Moreover, we go further in the comparison between Kalai and Ritz (1978)'s inseparability and Arrow (1963)'s single-peak restrictions, showing that the former condition is more "respectable," in the sense of Muller and Satterthwaite (1985).

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1 Introduction

Arrow (1963) established his celebrated impossibility theorem for Arrovian Social Welfare Functions (ASWFs) defined on the unrestricted domain of

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*This paper has been written to honor Nick Baigent and his distinguished contributions to social choice theory.

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preference orderings. As is well known, this result holds also for ASWFs defined on the domain of all antisymmetric preference orderings.

Kalai and Muller (1977) dealt with the problem of introducing restrictions on this latter domain of individual preferences in order to overcome Arrow's impossibility theorem.[1] They defined the notion of a decomposable domain, and provided the first complete characterization theorem, establishing that a domain of antisymmetric preference orderings admits ASWFs "without ties" - that is ASWFs which do not include indifference between distinct alternatives in the range - if and only if it is decomposable.

In a later unpublished paper, Kalai and Ritz (1978) introduced the notion of a domain containing an inseparable ordered pair. They showed that such a domain must be decomposable, and consequently it always admits nondictatorial ASWFs without ties. Domains containing an inseparable ordered pair were studied, among others, by Kalai and Ritz (1980), Kim and Roush ((1980),(1981)), Blair and Muller (1983), Ritz ((1983), (1985)), Muller and Satterthwaite (1985).

More recently, in two crucial papers, Sethuraman, Teo, and Vohra ((2003), (2006)) introduced the systematic use of integer programming in the traditional field of social choice theory, initiated by Arrow (1963). Following their approach, Busetto, Codognato, and Tonin (2012) reformulated Kalai and Muller (1977)'s characterization theorem; moreover, they introduced the notion of a strictly decomposable domain, and provided a new characterization theorem, establishing that a domain of antisymmetric preference orderings admits ASWFs "with ties" - that is ASWFs including indifference between distinct alternatives in the range - if and only if it is strictly decomposable. They showed that a strictly decomposable domain must be decomposable, whereas the converse relation does not hold.

In this paper, we use the integer programs formulated by Busetto et al. (2012) to study the relationship between domains of antisymmetric preference orderings containing an inseparable ordered pair and strictly decomposable domains. We show that a domain containing an inseparable ordered pair must be strictly decomposable, and consequently it always admits also ASWFs with ties; the converse relation does not hold. Moreover, we show that, when the set of alternatives is finite, a domain of single-peaked preference orderings à la Arrow must contain an inseparable ordered pair and, therefore, it is strictly decomposable. Drawing from Kim and Roush (1980) and Kalai and Satterthwaite (1985), we also deal with some questions related

[1]Maskin (1979) independently investigated the same issue.

to measuring how restrictive are inseparability and single-peak assumptions.

2 Notation and definitions

Let E be any initial finite subset of the natural numbers with at least two elements and let $|E|$ be the cardinality of E, denoted by n. Elements of E are called agents.

Let $\mathcal{E}$ be the collection of all subsets of E. Given a set $S \in \mathcal{E}$, let $S^c = E \setminus S$.

Let $\mathcal{A}$ be a set such that $|\mathcal{A}| \geq 3$. Elements of $\mathcal{A}$ are called alternatives.

Let $\mathcal{A}^2$ denote the set of all ordered pairs of alternatives.

Let $\mathcal{R}$ be the set of all the complete and transitive binary relations on $\mathcal{A}$, called preference orderings.

Let Σ be the set of all antisymmetric preference orderings.

Let Ω denote a nonempty subset of Σ. An element of Ω is called admissible preference ordering and is denoted by $\mathbf{p}$. We write $x\mathbf{p}y$ if x is ranked above y under $\mathbf{p}$.

Given $\mathbf{p} \in \Sigma$, let $\mathbf{p}^{-1}$ denote an antisymmetric preference ordering such that, for each $(x, y) \in \mathcal{A}^2$, $x\mathbf{p}y$ if and only if $y\mathbf{p}^{-1}x$

A pair $(x, y) \in \mathcal{A}^2$ is called trivial if there are not $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y$ and $y\mathbf{q}x$. Let TR denote the set of trivial pairs. We adopt the convention that all pairs $(x, x) \in \mathcal{A}^2$ are trivial.

A pair $(x, y) \in \mathcal{A}^2$ is nontrivial if it is not trivial. Let NTR denote the set of nontrivial pairs.

According to Kalai and Ritz (1978), Ω is said to contain an inseparable ordered pair if there exists $(u, v) \in NTR$ such that, for no $\mathbf{p} \in \Omega$ and $t \in \mathcal{A}$, $u\mathbf{p}t\mathbf{p}v$.

Let Ω^n denote the n-fold Cartesian product of Ω. An element of Ω^n is called a preference profile and is denoted by $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is the antisymmetric preference ordering of agent $i \in E$.

A Social Welfare Function (SWF) on Ω is a function $f : \Omega^n \to \mathcal{R}$.

f is said to be "without ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) = \emptyset$.

f is said to be "with ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) \neq \emptyset$.

Given $\mathbf{P} \in \Omega^n$, let $P(f(\mathbf{P}))$ and $I(f(\mathbf{P}))$ be binary relations on $\mathcal{A}$. We write $xP(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ but not $yf(\mathbf{P})x$ and $xI(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ and $yf(\mathbf{P})x$.

A SWF on Ω, f, satisfies Pareto Optimality (PO) if, for all $(x, y) \in \mathcal{A}^2$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_iy$, for all $i \in E$, implies $xP(f(\mathbf{P}))y$.

A SWF on Ω, f, satisfies Independence of Irrelevant Alternatives (IIA) if, for all $(x, y) \in NTR$ and for all $\mathbf{P}, \mathbf{P}' \in \Omega^n$, $x\mathbf{p}_i y$ if and only if $x\mathbf{p}'_i y$, for all $i \in E$, implies, $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P}')y$, and, $yf(\mathbf{P})x$ if and only if $yf(\mathbf{P}')x$.

An Arrovian Social Welfare Function (ASWF) on Ω is a SWF on Ω, f, which satisfies PO and IIA.

An ASWF on Ω, f, is dictatorial if there exists $j \in E$ such that, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_j y$ implies $xP(f(\mathbf{P}))y$. f is nondictatorial if it is not dictatorial.

Given $(x, y) \in \mathcal{A}^2$ and $S \in \mathcal{E}$, let $d_S(x, y)$ denote a variable such that $d_S(x, y) \in \{0, \frac{1}{2}, 1\}$.

An Integer Program (IP) on Ω consists of a set of linear constraints, related to the preference orderings in Ω, on variables $d_S(x, y)$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, and of the further conventional constraints that $d_E(x, y) = 1$ and $d_\emptyset(y, x) = 0$, for all $(x, y) \in TR$.

Let d denote a feasible solution (henceforth, for simplicity, only "solution") to an IP on Ω. d is said to be a binary solution if variables $d_S(x, y)$ reduce to assume values in the set $\{0, 1\}$, for all $(x, y) \in NTR$, and for all $S \in \mathcal{E}$. It is said to be a "ternary" solution, otherwise.

A solution d is dictatorial if there exists $j \in E$ such that $d_S(x, y) = 1$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, with $j \in S$. d is nondictatorial if it is not dictatorial.

An ASWF on Ω, f, and a solution to an IP on the same Ω, d, are said to correspond if, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, $xP(f(\mathbf{P}))y$ if and only if $d_S(x, y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x, y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$.

We define now the notion of decomposability, introduced by Kalai and Muller (1977) to characterize the domains of antisymmetric preference orderings admitting nondictatorial ASWFs without ties, and the notion of strict decomposability, introduced by Busetto et al. (2012) to characterize the domains of antisymmetric preference orderings admitting nondictatorial ASWFs with ties.

Consider a set $R \subset \mathcal{A}^2$. Consider the following conditions on R.

Condition 1. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R$ implies that $(x, z) \in R$.

Condition 2. For all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$.

Condition 3. *There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x, y) \in R^*$ implies that $(x, z) \in R$.*

Condition 4. *There exists a set $R^* \subset \mathcal{A}^2$, with $R \cap R^* = \emptyset$, such that, for all triples of alternatives x, y, z, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, then $(x, y) \in R$ and $(y, z) \in R^*$ imply that $(x, z) \in R$, and $(x, y) \in R^*$ and $(y, z) \in R$ imply that $(x, z) \in R$.*

A domain Ω is said to be decomposable if and only if there exist two sets R_1 and R_2, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, R_i, $i = 1, 2$, satisfies Conditions 1 and 2.

A domain Ω is said to be strictly decomposable if and only if there exist four sets R_1, R_2, R_1^*, and R_2^*, with $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$; $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$; moreover, R_i, $i = 1, 2$, satisfies Condition 1; R_i and R_i^*, $i = 1, 2$, satisfy Condition 2; each pair (R_i, R_i^*), $i = 1, 2$, satisfies Conditions 3 and 4.

3 Domains containing an inseparable ordered pair and nondictatorial ASWFs

Kalai and Ritz (1978) contains the first investigation of the relationship between their notion of a domain containing an inseparable ordered pair and Kalai and Muller (1977)'s notion of a decomposable domain. In this section, we will extend the analysis to the relationship between the notion of a domain containing an inseparable ordered pair and Busetto et al. (2012)'s notion of a strictly decomposable domain. We will follow the approach - initiated by Sethuraman et al. (2003) - which systematically applies integer programming tools to social choice theory. In particular, we will use a "ternary" IP on Ω proposed by Busetto et al. (2012). According to this work, we will call it IP1$'$. It consists of the following set of constraints:

$$d_E(x, y) = 1, \tag{1}$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \tag{2}$$

for all $(x,y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_S(x,y) \leq d_S(x,z), \tag{3}$$

if $d_S(x,y) \in \{0,1\}$;

$$d_S(x,y) < d_S(x,z), \tag{4}$$

if $d_s(x,y) = \frac{1}{2}$, for all triples x,y,z such that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S \in \mathcal{E}$;

$$d_S(x,y) + d_S(y,z) \leq 1 + d_S(x,z), \tag{5}$$

if $d_s(x,y), d_s(y,z) \in \{0,1\}$;

$$d_S(x,y) + d_S(y,z) = \frac{1}{2} + d_S(x,z), \tag{6}$$

if $d_S(x,y) = \frac{1}{2}$ or $d_S(y,z) = \frac{1}{2}$, for all triples x,y,z such that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, and for all $S \in \mathcal{E}$.

On the basis of IP1$'$, we state and prove now our main result.

Theorem. *If Ω contains an inseparable ordered pair, then there exists a nondictatorial ternary solution to IP1$'$ on Ω, d, for $n = 2$.*

Proof. Suppose that Ω contains an inseparable ordered pair $(u,v) \in NTR$. For each $(x,y) \in NTR$, let $d_\emptyset(x,y) = 0$, $d_E(x,y)$=1. Moreover, let $d_{\{1\}}(x,y)$ = 1 and $d_{\{2\}}(y,x) = 0$, if $(x,y) \neq (u,v)$; $d_{\{1\}}(x,y) = \frac{1}{2}$ and $d_{\{2\}}(y,x) = \frac{1}{2}$, if $(x,y) = (u,v)$. Then, d satisfies (1) and (2). Consider a triple x,y,z. Suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x,y) \in \{0,1\}$ and

$$d_{\{1\}}(x,y) > d_{\{1\}}(x,z).$$

Then, $(x,z) = (u,v)$. But then, (u,v) is not inseparable as $u\mathbf{p}y\mathbf{p}v$, a contradiction. Now, suppose that $d_{\{2\}}(x,y) \in \{0,1\}$ and

$$d_{\{2\}}(x,y) > d_{\{2\}}(x,z).$$

Then, we have $d_{\{2\}}(x,y) = 1$, a contradiction. Therefore, d satisfies (3). Suppose that $d_{\{1\}}(x,y) = \frac{1}{2}$ and

$$d_{\{1\}}(x,y) \geq d_{\{1\}}(x,z).$$

Then, we have $(x, y) = (u, v)$. But then, we have $d_{\{1\}}(x, z) = 1$, a contradiction. Suppose that $d_{\{2\}}(x, y) = \frac{1}{2}$ and

$$d_{\{2\}}(x, y) \geq d_{\{2\}}(x, z).$$

Then, we have $(x, y) = (v, u)$. But then, (u, v) is not inseparable as $u\mathbf{p}z\mathbf{p}v$, a contradiction. Therefore, d satisfies (4). Consider a triple x, y, z and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y), d_{\{1\}}(y, z) \in \{0, 1\}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z).$$

Then, we have $(x, z) = (u, v)$. But then, (u, v) is not inseparable as $u\mathbf{p}y\mathbf{p}v$, a contradiction. Now, suppose that $d_{\{2\}}(x, y), d_{\{2\}}(y, z) \in \{0, 1\}$ and

$$d_{\{2\}}(x, y) + d_{\{2\}}(y, z) > 1 + d_{\{2\}}(x, z).$$

Then, we have $d_{\{2\}}(x, y) = 1$ and $d_{\{2\}}(y, z) = 1$, a contradiction. Therefore, d satisfies (5). Suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > \frac{1}{2} + d_{\{1\}}(x, z).$$

Then, we have $d_{\{1\}}(x, z) = 0$, a contradiction. Suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) < \frac{1}{2} + d_{\{1\}}(x, z).$$

Then, we have $d_{\{1\}}(y, z) = 0$, a contradiction. Suppose that $d_{\{2\}}(x, y) = \frac{1}{2}$ and

$$d_{\{2\}}(x, y) + d_{\{2\}}(y, z) > \frac{1}{2} + d_{\{2\}}(x, z).$$

Then, we have $d_{\{2\}}(y, z) = 1$, a contradiction. Suppose that $d_{\{2\}}(x, y) = \frac{1}{2}$ and

$$d_{\{2\}}(x, y) + d_{\{2\}}(y, z) < \frac{1}{2} + d_{\{2\}}(x, z).$$

Then, we have $d_{\{2\}}(x, z) = 1$, a contradiction. Therefore, d satisfies (6). Hence, d is a nondictatorial ternary solution to IP1′ on Ω. ■

In their Theorem 5, Busetto et al. (2012) showed that there exists a nondictatorial ternary solution to IP1′ on Ω if and only if Ω is strictly decomposable. By exploiting this result, we establish here the relationship between the notions of a domain containing an inseparable ordered pair and of a strictly decomposable domain as a corollary to our Theorem.

Corollary 1. *If Ω contains an inseparable ordered pair, then it is strictly decomposable.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, there exists a nondictatorial ternary solution to IP1$'$ on Ω, d, for $n = 2$, by our main theorem. Bur then, Ω is strictly decomposable, by Theorem 5 in Busetto et al. (2012). ■

The following example shows that the converse of Corollary 1 does not hold.

Example 1. *Let A be the closed interval $[0, 1]$ of the real line and $\Omega = \{\mathbf{p}, \mathbf{p}^{-1}\}$, where $\mathbf{p}$ is such that, if $x, y \in [0, 1]$ and $x > y$, then $x\mathbf{p}y$. Then, Ω is strictly decomposable but it does not contain an inseparable ordered pair.*

Proof. Let $V_i = \emptyset$, $i = 1, 2$, $V_1^* = \{(x, y) \in NTR : x\mathbf{p}y\}$, $V_2^* = \{(x, y) \in NTR : x\mathbf{p}^{-1}y\}$. Then, we have $\emptyset \subsetneqq V_i^* \subset NTR$, $i = 1, 2$. Moreover, for all $(x, y) \in NTR$, we have $(x, y) \in V_1^*$ if and only if $(y, x) \in V_2^*$. Finally, V_i^*, $i = 1, 2$, satisfies Condition 2. Therefore, Ω is strictly decomposable. Moreover, it is straightforward to verify that Ω does not contain an inseparable ordered pair. ■

Busetto et al. (2012) also proved, in their Corollary 5, that there exists a nondictatorial ASWF with ties on a domain Ω, for any $n \geq 2$, if and only if Ω is strictly decomposable. On the basis of this result, we obtain a further corollary of our Theorem, establishing that a domain Ω which contains an inseparable ordered pair always admits a nondictatorial ASWF with ties, for $n \geq 2$.

Corollary 2. *If Ω contains an inseparable ordered pair, then there exists a nondictatorial ASWF with ties on Ω, f, for $n \geq 2$.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, it is strictly decomposable, by Corollary 1. But then, there exists a nondictatorial ASWF with ties on Ω, f, for $n \geq 2$, by Corollary 5 in Busetto et al. (2012). ■

Kalai and Ritz (1978)'s main result, establishing the relationship between the notions of a domain containing an inseparable ordered pair and of a decomposable domain, is here straightforwardly obtained as a corollary of our Theorem. The proof is based on Corollary 1 above and Busetto et al. (2012)'s Theorem 7. This last result establishes that a strictly decomposable domain is always decomposable.

Corollary 3. *If Ω contains an inseparable ordered pair, then it is decomposable.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, it is strictly decomposable, by Corollary 1. But then, it is decomposable, by Theorem 7 in Busetto et al. (2012). ■

The converse of Corollary 3 does not hold. This in an immediate implication of Busetto et al. (2012)'s Example 2, showing that a decomposable domain may not be strictly decomposable, and of our Corollary 1.

Instead, Corollary 3 has the following implication, which concerns the existence of ASWFs without ties on domains containing an inseparable ordered pair.

Corollary 4. *If Ω contains an inseparable ordered pair, then there exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$.*

Proof. Suppose that Ω contains an inseparable ordered pair. Then, it is decomposable, by Corollary 3. But then, there exists a nondictatorial ASWF without ties on Ω, f, for $n \geq 2$, as is well known by Kalai and Muller (1977)'s characterization theorem.[2] ■

The next result concludes the analysis - in terms of integer programming - of the relationships of the notion of a domain containing an inseparable ordered pair with those of a strictly decomposable domain, and a decomposable domain.

Corollary 5. *If Ω is decomposable but not strictly decomposable, then it does not contain an inseparable ordered pair.*

Proof. It is a straightforward consequence of our Theorem. ■

Muller and Satterthwaite (1985) dealt with the question of measuring how restrictive are the various conditions imposed on a domain Ω in order to make it to admit nondictatorial ASWFs. They suggested to use the ratio $\frac{|\Omega|}{|\Sigma|}$ to evaluate whether "the size of [a] restricted domain is still "respectable" relative to the size of the full domain." They considered that a "respectable" relative size should provide "an indication that [...] characterizations are not very restrictive" (see pp. 154-155).

The remainder of this section is devoted to show that there exist decomposable and strictly decomposable domains whose size is "respectable," if

[2] See also Corollary 2 in Busetto et al. (2012), for a version of this theorem in terms of integer programming.

compared to the size of the full domain, in that they contain more than half of all possible preference orderings.

In order to obtain these results, we first need to show the following proposition, establishing that the minimal cardinality of decomposable and strictly decomposable domains is $|\Omega| = 2$.

Proposition 1. $\min |\Omega| = 2$, *either for all decomposable* $\Omega \in \Sigma$ *and for all strictly decomposable* $\Omega \in \Sigma$.

Proof. Consider $\Omega \in \Sigma$ and suppose that $|\Omega| = 1$. Then, Ω is neither decomposable nor strictly decomposable as $NTR = \emptyset$. Consider $\mathbf{p} \in \Sigma$ and suppose that $\Omega = \{\mathbf{p}, \mathbf{p}^{-1}\}$. Let $R_i = \{(x, y) \in NTR : x\mathbf{p}y\}$, $i = 1, 2$. Then, we have $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$. Moreover, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$. Finally, R_i, $i = 1, 2$, satisfies Condition 2. We have shown that Ω is decomposable. It can be easily shown that Ω is also strictly decomposable by using, in this more general context, the same argument as in the proof of Example 1. Hence, $\min |\Omega| = 2$, either for all decomposable $\Omega \in \Sigma$ and for all strictly decomposable $\Omega \in \Sigma$. ■

In what follows, we will also exploit the following results due to Kim and Rousch (1980).

Consider an ordered pair $(u, v) \in \mathcal{A}^2$. Let $\Omega_{(u,v)} = \{\mathbf{p} \in \Sigma : u\mathbf{p}t\mathbf{p}v,$ for some $t \in \mathcal{A}\}$. Moreover, let $\Omega^*_{(u,v)} = \Sigma \setminus \Omega_{(u,v)}$. In their Theorem 5.2.7, Kim and Rousch (1980) showed that, if $|\mathcal{A}| = m$, $|\Omega^*_{(u,v)}| = \frac{m!}{2} + (m - 1)!$. Then, in their Theorem 5.2.8, these authors showed that the domains containing exactly one inseparable ordered pair, whose cardinality is that established in Theorem 5.2.7, are the largest nondictatorial domains.

Now, let $H = \{q \in Q_+ : q = \frac{|\Omega|}{|\Sigma|}$, for some decomposable $\Omega\}$, where Q_+ is the set of positive rational numbers. Our first result says that, if the set of alternatives is finite, the set H has a maximum and a minimum.

Proposition 2. *Let* $|\mathcal{A}| = m$. $\max H = \frac{1}{2} + \frac{1}{m}$ *and* $\min H = \frac{2}{m!}$.

Proof. $\Omega^*_{(u,v)}$ is decomposable, by Corollary 3. Then, $\frac{1}{2} + \frac{1}{m} \in H$ as $\frac{|\Omega^*_{(u,v)}|}{|\Sigma|} = \frac{1}{2} + \frac{1}{m}$. $|\Omega| \leq \frac{m!}{2} + (m-1)!$, for all decomposable $\Omega \in \Sigma$, by Theorem 5.2.8 in Kim and Roush (1980). Then, $\max H = \frac{1}{2} + \frac{1}{m}$. $\min |\Omega| = 2$, for all decomposable $\Omega \in \Sigma$, by Proposition 1. Then, $\min H = \frac{2}{m!}$. ■

Let $K = \{q \in Q_+ : q = \frac{|\Omega|}{|\Sigma|}$, for some strictly decomposable $\Omega\}$. The following proposition shows that also the set K has a maximum and a minimum.

Proposition 3. *Let* $|\mathcal{A}| = m$. $\max K = \frac{1}{2} + \frac{1}{m}$ *and* $\min K = \frac{2}{m!}$

Proof. $\Omega^*_{(u,v)}$ is strictly decomposable, by our Theorem. Then, $\frac{1}{2} + \frac{1}{m} \in K$ as $\frac{|\Omega^*_{(u,v)}|}{|\Sigma|} = \frac{1}{2} + \frac{1}{m}$. $|\Omega| \leq \frac{m!}{2} + (m-1)!$, for all strictly decomposable $\Omega \in \Sigma$, by Theorem 5.2.8 in Kim and Roush (1980), as each strictly decomposable domain is decomposable, by Theorem 7 in Busetto et al. (2012). Then, $\max K = \frac{1}{2} + \frac{1}{m}$. $\min |\Omega| = 2$, for all strictly decomposable $\Omega \in \Sigma$, by Proposition 1. Then, $\min K = \frac{2}{m!}$. ■

4 Domains of single-peaked preference orderings à la Arrow

The specific purpose of Kalai and Ritz (1978) was "to show that the single-peak condition (see Black (1948) and Arrow (1963)) is a special case of a simpler condition of inseparability" (see p. 1). Black (1948) and Arrow (1963) introduced slightly different definitions of a single-peaked preference ordering. In particular, with a view to overcome his impossibility theorem, Arrow (1963) proposed a notion of single-peakedness which extends the one previously introduced by Black (1948). In their Example 1, Kalai and Ritz (1978) employed the notion of a single-peaked preference ordering à la Arrow, which we remind here.

Definition. *Given* $\mathbf{q} \in \Sigma$, $\mathbf{p} \in \Sigma$ *is said to be single-peaked à la Arrow relative to* $\mathbf{q}$ *if, for all alternatives* $x, y, z \in A$, $x\mathbf{q}y\mathbf{q}z$ *and* $x\mathbf{p}y$ *implies* $y\mathbf{p}z$.

Given $\mathbf{q} \in \Sigma$, we denote by $\Omega_{\mathbf{q}}$ the set of all preference orderings which are single-peaked à la Arrow relative to $\mathbf{q}$.

The following proposition extends a statement contained in Kalai and Ritz (1978)'s Example 1.

Proposition 4. *Given* $\mathbf{q} \in \Sigma$, *if there are* $u, v \in \mathcal{A}$ *such that* $u\mathbf{q}v$, $u\mathbf{q}z$, $v\mathbf{q}z$, *for all* $z \in \mathcal{A} \setminus \{u, v\}$, *or* $v\mathbf{q}u$, $z\mathbf{q}u$, $z\mathbf{q}v$, *for all* $z \in \mathcal{A} \setminus \{u, v\}$, *then the ordered pair* (u, v) *is inseparable in* $\Omega_{\mathbf{q}}$.

Proof. Consider $\mathbf{q} \in \Sigma$. Suppose that there are $u, v \in \mathcal{A}$ such that $u\mathbf{q}v$. Moreover, suppose that $u\mathbf{q}z$, $v\mathbf{q}z$, for all $z \in \mathcal{A} \setminus \{u, v\}$. Suppose that the ordered pair (u, v) is not inseparable in $\Omega_{\mathbf{q}}$. Then, there are $t \in \mathcal{A}$ and $\mathbf{p} \in \Omega_{\mathbf{q}}$ such that $u\mathbf{p}t\mathbf{p}v$. But we must also have that $v\mathbf{p}t$, since $u\mathbf{q}v\mathbf{q}t$, $u\mathbf{p}v$, and $\mathbf{p}$ is single-peaked à la Arrow relative to $\mathbf{q}$, a contradiction. The case where $v\mathbf{q}u$, $z\mathbf{q}u$, $z\mathbf{q}v$, for all $z \in \mathcal{A} \setminus \{u, v\}$ leads, *mutatis mutandis*, to the same contradiction. ■

In their Example 2, Kalai and Muller (1977) showed that, for any $\mathbf{q} \in \Sigma$, $\Omega_{\mathbf{q}}$ is decomposable. Our next proposition shows that, if the set of alternatives if finite, $\Omega_{\mathbf{q}}$ must also contain an inseparable ordered pair.

Proposition 5. *Let $|\mathcal{A}| = m$. For each $\mathbf{q} \in \Sigma$, $\Omega_{\mathbf{q}}$ contains an inseparable ordered pair.*

Proof. There are $u, v \in \mathcal{A}$ such that $u\mathbf{q}v$, $u\mathbf{q}z$, $v\mathbf{q}z$, for all $z \in \mathcal{A} \setminus \{u, v\}$ as $|\mathcal{A}| = m$. Consider $\mathbf{q} \in \Sigma$. Suppose that $\Omega_{\mathbf{q}}$ does not contain an inseparable pair. Then, there are not $u, v \in \mathcal{A}$ such that $u\mathbf{q}v$, $u\mathbf{q}z$, $v\mathbf{q}z$, for all $z \in \mathcal{A} \setminus \{u, v\}$, by the contrapositive of Proposition 4, a contradiction. Hence, $\Omega_{\mathbf{q}}$ contains an inseparable ordered pair. ■

The following example adapts to our context Kalai and Ritz (1978)'s Example 2, and shows that the converse of Proposition 5 does not hold.

Example 2. *Let $|\mathcal{A}| = 3$. Then, $\Omega^*_{(u,v)}$ is a domain which contains an inseparable ordered pair but which is such that $\Omega^*_{(u,v)} \neq \Omega_q$, for each $\mathbf{q} \in \Sigma$.*

Proof. Let $\mathcal{A} = \{a, b, c\}$. Consider, without loss of generality, the ordered pair (a, c). Then, $\Omega^*_{(a,c)} = \{\mathbf{p} \in \Sigma : a\mathbf{p}c\mathbf{p}b, b\mathbf{p}a\mathbf{p}c, b\mathbf{p}c\mathbf{p}a, c\mathbf{p}a\mathbf{p}b, c\mathbf{p}b\mathbf{p}a\}$. But then, it is straightforward to verify that $\Omega^*_{(a,c)} \neq \Omega_q$, for each $\mathbf{q} \in \Sigma$. ■

As already mentioned, by employing his single-peak condition, Arrow (1963) could circumvent his impossibility theorem. In Muller and Satterthwaite (1985)'s vein, we deal with the question of measuring how restrictive is this condition, compared with Kalai and Ritz (1978)'s inseparability condition. By Kim and Roush (1980)'s Theorem 5.2.7, mentioned in Section 3, we know that $|\Omega_q| = 2^{m-1}$, for each $\mathbf{q} \in \Sigma$. This implies that the relative size of the domains of single-peaked preference orderings, $\frac{|\Omega_q|}{|\Sigma|} = \frac{2^{m-1}}{m!}$, is less "respectable" than the relative size of the domains containing exactly one inseparable ordered pair, $\frac{|\Omega^*_{(u,v)}|}{|\Sigma|} = \frac{1}{2} + \frac{1}{m}$. In particular, when $m > 3$, a domain of single-peaked preference orderings à la Arrow contains less than half of all possible preference orderings. This shows that Arrow's impossibility theorem can be circumvented with a minor "waste" of preference orderings by imposing Kalai and Ritz (1978)'s inseparability condition rather than the single-peak restriction à la Arrow.

The following corollary to Proposition 5 shows that, with a finite number of alternatives, Ω_q must be strictly decomposable.

Corollary 6. *Let $|\mathcal{A}| = m$. For each $\mathbf{q} \in \Sigma$, $\Omega_{\mathbf{q}}$ is strictly decomposable.*

Proof. Consider $\mathbf{q} \in \Sigma$. Suppose that $\Omega_{\mathbf{q}}$ is not strictly decomposable. Then, it does not contain an inseparable ordered pair, by the contrapositive of Corollary 1, contradicting Proposition 5. ■

To conclude, let us notice that Proposition 5 and the two related results we have presented in this section no longer hold if the assumption that the set of alternative if finite is removed. To show it, suppose that $\mathcal{A}$ and $\mathbf{p}$ are as in Example 1. Then, it is straightforward to verify that $\Omega_{\mathbf{p}}$ does not contain an inseparable ordered pair.

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