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# Integer Programming and Nondictatorial Arrovian Social Welfare Functions

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**Abstract**

Following Sethuraman, Teo and Vohra ((2003), (2006)), we apply integer programming tools to the analysis of fundamental issues in social choice theory. We generalize Sethuraman et al.'s approach specifying integer programs in which variables are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$. We show that there exists a one-to-one correspondence between the solutions of an integer program defined on this set and the set of the Arrovian social welfare functions with ties (i.e. admitting indifference in the range). We use our generalized integer programs to analyze nondictatorial Arrovian social welfare functions, in the line opened by Kalai and Muller (1977). Our main theorem provides a complete characterization of the domains admitting nondictatorial Arrovian social welfare functions with ties by introducing a notion of strict decomposability.

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## 1 Introduction

In two pathbreaking papers, Sethuraman, Teo, and Vohra ((2003), (2006)) introduced the systematic use of integer programming in the traditional field of social choice theory, initiated by Arrow (1963). As remarked by these

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authors, integer programming is a powerful analytical tool, which makes it possible to derive, in a systematic and simple way, many of the already known theorems on Arrovian Social Welfare Functions (ASWFs) - that is those social welfare functions satisfying the hypotheses of Pareto optimality and independence of irrelevant alternatives - and to prove new results.

In particular, they developed Integer Programs (IPs) in which variables assume values only in the set $\{0, 1\}$. Binary IPs of this kind are suitable to be used as an auxiliary tool to represent the so-called ASWFs "without ties," that is ASWFs which do not admit indifference between distinct alternatives in their range. Indeed, a fundamental theorem in Sethuraman et al. (2003) establishes a one-to-one correspondence, on domains of antisymmetric preference orderings, between the set of feasible solutions of their main binary IP and the set of ASWFs without ties. In both papers mentioned above, Sethuraman et al. used binary integer programming to analyze, among other issues, neutral and anonymous ASWFs. Moreover, in the 2003 paper, they opened the way to a reconsideration, in terms of integer programming, of the work by Kalai and Muller (1977) on nondictatorial ASWFs.

Arrow (1963) established his celebrated impossibility theorem for ASWFs defined on the unrestricted domain of preference orderings. As is well known, this result holds also for ASWFs defined on the domain of all antisymmetric preference orderings. Kalai and Muller (1977) dealt with the problem of introducing restrictions on this latter domain of individual preferences in order to overcome Arrow's impossibility result.[1] They gave the first complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs without ties. They did this by means of two theorems. In their Theorem 1, they showed that there exists a $n$-person nondictatorial ASWF for a given domain of antisymmetric preference orderings if and only if there exists a 2-person nondictatorial ASWF for the same domain. In their Theorem 2, they gave the domain characterization, by introducing the concept of decomposability.

Sethuraman et al. (2003) provided a simplified version of Kalai and Muller's Theorem 1 by using a binary IP.

In this paper, we proceed along the way opened by Sethuraman et al. We provide a natural generalization of their approach, specifying IPs in which variables are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$. These programs - which we will call "ternary IPs," with some abuse with respect to the current

[1] Maskin (1979) independently investigated the same issue.

specialized literature [2] - are suitable to be used to represent ASWFs "with ties" - that is ASWFs which admit indifference between distinct alternatives in their range. Indeed, we provide a theorem establishing that there exists a one-to-one correspondence between the set of feasible solutions of a ternary IP and the set of ASWFs with ties.

We use our generalized integer programs to systematically study nondictatorial ASWFs. We first show how these tools can be used to obtain a new and simpler proof of Kalai and Muller's Theorem 2 for ASWFs without ties. To this end, we introduce a simpler but equivalent version of the concept of decomposability proposed by these authors. More important, this analysis is the basis for going beyond the already known results on nondictatorial ASWFs.

In fact, Kalai and Muller's Theorem 2 provides a complete characterization both of the domains of antisymmetric preference orderings admitting nondictatorial ASWFs without ties and those admitting dictatorial ASWFs without ties. The problem of characterizing the domains of antisymmetric preference orderings admitting nondictatorial ASWFs with ties has so far been left open. We overcome this problem by using ternary integer programming: in our main theorem, we provide a complete characterization of these domains by introducing the notion of strict decomposability.

This new characterization result raises the question of which is the relationship between decomposable and strictly decomposable domains. We conclude our analysis showing that all strictly decomposable domains are decomposable whereas the converse relation does not hold.

## 2 Notation and definitions

Let $E$ be any initial finite subset of the natural numbers with at least two elements and let $|E|$ be the cardinality of $E$, denoted by $n$. Elements of $E$ are called agents.

Let $\mathcal{E}$ be the collection of all subsets of $E$. Given a set $S \in \mathcal{E}$, let $S^c = E \setminus S$.

Let $\mathcal{A}$ be a set such that $|\mathcal{A}| \geq 3$. Elements of $\mathcal{A}$ are called alternatives.

Let $\mathcal{A}^2$ denote the set of all ordered pairs of alternatives.

[2] We have to stress that we still apply the basic tools of integer linear programming and that the programs we introduce could be equivalently defined on the set $\{0, 1, 2\}$. Nonetheless, here we prefer to follow Sethuraman et al. (2006), and keep using the value $\frac{1}{2}$ in order to incorporate indifference between social alternatives into the analysis.

Let $\mathcal{R}$ be the set of all the complete and transitive binary relations on $\mathcal{A}$, called preference orderings.

Let $\Sigma$ be the set of all antisymmetric preference orderings.

Let $\Omega$ denote a nonempty subset of $\Sigma$. An element of $\Omega$ is called admissible preference ordering and is denoted by $\mathbf{p}$. We write $x\mathbf{p}y$ if $x$ is ranked above $y$ under $\mathbf{p}$.

A pair $(x, y) \in \mathcal{A}^2$ is called trivial if there are not $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y$ and $y\mathbf{q}x$. Let $TR$ denote the set of trivial pairs. We adopt the convention that all pairs $(x, x) \in \mathcal{A}^2$ are trivial.

A pair $(x, y) \in \mathcal{A}^2$ is nontrivial if it is not trivial. Let $NTR$ denote the set of nontrivial pairs.

Let $\Omega^n$ denote the $n$-fold Cartesian product of $\Omega$. An element of $\Omega^n$ is called a preference profile and is denoted by $\mathbf{P} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, where $\mathbf{p}_i$ is the antisymmetric preference ordering of agent $i \in E$.

A Social Welfare Function (SWF) on $\Omega$ is a function $f : \Omega^n \to \mathcal{R}$.

$f$ is said to be "without ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) = \emptyset$.

$f$ is said to be "with ties" if $f(\Omega^n) \cap (\mathcal{R} \setminus \Sigma) \neq \emptyset$.

Given $\mathbf{P} \in \Omega^n$, let $P(f(\mathbf{P}))$ and $I(f(\mathbf{P}))$ be binary relations on $\mathcal{A}$. We write $xP(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ but not $yf(\mathbf{P})x$ and $xI(f(\mathbf{P}))y$ if, for $x, y \in \mathcal{A}$, $xf(\mathbf{P})y$ and $yf(\mathbf{P})x$.

A SWF on $\Omega$, $f$, satisfies Pareto Optimality (PO) if, for all $(x, y) \in \mathcal{A}^2$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_iy$, for all $i \in E$, implies $xP(f(\mathbf{P}))y$.

A SWF on $\Omega$, $f$, satisfies Independence of Irrelevant Alternatives (IIA) if, for all $(x, y) \in NTR$ and for all $\mathbf{P}, \mathbf{P}' \in \Omega^n$, $x\mathbf{p}_iy$ if and only if $x\mathbf{p}'_iy$, for all $i \in E$, implies, $xf(\mathbf{P})y$ if and only if $xf(\mathbf{P}')y$, and, $yf(\mathbf{P})x$ if and only if $yf(\mathbf{P}')x$.

An Arrovian Social Welfare Function (ASWF) on $\Omega$ is a SWF on $\Omega$, $f$, which satisfies PO and IIA.

An ASWF on $\Omega$, $f$, is dictatorial if there exists $j \in E$ such that, for all $(x, y) \in NTR$ and for all $\mathbf{P} \in \Omega^n$, $x\mathbf{p}_jy$ implies $xP(f(\mathbf{P}))y$. $f$ is nondictatorial if it is not dictatorial.

Given $(x, y) \in \mathcal{A}^2$ and $S \in \mathcal{E}$, let $d_S(x, y)$ denote a variable such that $d_S(x, y) \in \{0, \frac{1}{2}, 1\}$.

An Integer Program (IP) on $\Omega$ consists of a set of linear constraints, related to the preference orderings in $\Omega$, on variables $d_S(x, y)$, for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$, and of the further conventional constraints that $d_E(x, y) = 1$ and $d_\emptyset(y, x) = 0$, for all $(x, y) \in TR$.

Let $d$ denote a feasible solution (henceforth, for simplicity, only "solution") to an IP on $\Omega$. $d$ is said to be a binary solution if variables $d_S(x, y)$

reduce to assume values in the set $\{0,1\}$, for all $(x,y) \in NTR$, and for all $S \in \mathcal{E}$. It is said to be a "ternary" solution, otherwise.

A solution $d$ is dictatorial if there exists $j \in E$ such that $d_S(x,y) = 1$, for all $(x,y) \in NTR$ and for all $S \in \mathcal{E}$, with $j \in S$. $d$ is nondictatorial if it is not dictatorial.

An ASWF on $\Omega$, $f$, and a solution to an IP on the same $\Omega$, $d$, are said to correspond if, for each $(x,y) \in NTR$ and for each $S \in \mathcal{E}$, $xP(f(\mathbf{P}))y$ if and only if $d_S(x,y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x,y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x,y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$.

Finally, consider the following condition of decomposability, introduced by Kalai and Muller (1977) to characterize the domains of antisymmetric preference orderings admitting nondictatorial ASWFs without ties.

$\Omega$ is said to be decomposable (henceforth, KM decomposable) if there exists a set $R$, with $TR \subsetneqq R \subsetneqq \mathcal{A}^2$, satisfying the following conditions.

*Condition I. For every two pairs $(x,y), (x,z) \in NTR$, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(x,y) \in R$ implies that $(x,z) \in R$.*

*Condition II. For every two pairs $(x,y), (x,z) \in NTR$, if there exist $\mathbf{p}, \mathbf{q} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then $(z,x) \in R$ implies that $(y,x) \in R$.*

*Condition III. For every two pairs $(x,y), (x,z) \in NTR$, if there exists $\mathbf{p} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$, then $(x,y) \in R$ and $(y,z) \in R$ imply that $(x,z) \in R$.*

*Condition IV. For every two pairs $(x,y), (x,z) \in NTR$, if there exists $\mathbf{p} \in \Omega$ for which $x\mathbf{p}y\mathbf{p}z$, then $(z,x) \in R$ implies that $(y,x) \in R$ or $(z,y) \in R$.*

# 3 Binary integer programming and Arrovian social welfare functions without ties: the work of Sethuraman, Teo and Vohra

The first formulation of an IP on $\Omega$ was proposed by Sethuraman et al. (2003), for the case where $d_S(x,y) \in \{0,1\}$, for all $(x,y) \in NTR$ and for all $S \in \mathcal{E}$. This binary IP - which we will call IP0 - consists of the following set of constraints:

$$d_E(x,y) = 1, \qquad (i)$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \qquad (ii)$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \qquad (iii)$$

for all triples of alternatives $x, y, z$ and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy the following conditions (hereafter referred to as Conditions $(*)$):

$$\begin{aligned}
A &\neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}z\mathbf{p}y, \\
B &\neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}x\mathbf{p}z, \\
C &\neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}y\mathbf{p}x, \\
U &\neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } x\mathbf{p}y\mathbf{p}z, \\
V &\neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } z\mathbf{p}x\mathbf{p}y, \\
W &\neq \emptyset \text{ only if there exists } \mathbf{p} \in \Omega \text{ such that } y\mathbf{p}z\mathbf{p}x.
\end{aligned}$$

By introducing integer programming, Sethuraman et al. (2003) were able to provide a new representation of ASWFs with respect to the axiomatic one previously used in the Arrow's tradition. In particular, in the 2003 paper, they showed that there exists a one-to-one correspondence between the set of the solutions to IP0 on $\Omega$ and the set of the ASWFs without ties on the same $\Omega$. Moreover, in both their 2003 and 2006 papers, they used IP0 to systematically analyze properties of ASWFs such as neutrality and anonymity.

Sethuraman et al. (2003) also built up a second binary IP on $\Omega$, for many respects related to Kalai and Muller's work on nondictatorial ASWFs. In this IP - which we will call IP0$'$ - constraint (iii) is replaced by the following set of constraints:

$$d_S(x, y) \leq d_S(x, z), \qquad (iv)$$

$$d_S(z, x) \leq d_S(y, x), \qquad (v)$$

for all triples $x, y, z$ such that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S \in \mathcal{E}$;

$$d_S(x, y) + d_S(y, z) \leq 1 + d_S(x, z), \qquad (vi)$$

$$d_S(z, y) + d_S(y, x) \geq d_S(z, x), \qquad (vii)$$

for all triples $x, y, z$ such that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and for all $S \in \mathcal{E}$.

Constraints (iv) and (v) translate, in terms of variables $d_S(x, y)$, Kalai and Muller's Conditions I and II. In their Claim 1, these authors showed that these constraints are special cases of (iii). Constraints (vi) and (vii) translate Kalai and Muller's Conditions III and IV. In their Claim 2, Sethuraman et al. (2003) showed that also these constraints are special cases of (iii). Their analysis established that any solution $d$ to IP0 on $\Omega$ is a solution to IP0$'$ on the same domain and that IP0 and IP0$'$ are equivalent in the case where $n = 2$.

In the remainder of this section, we will prove that the set of constraints (iv)-(vii) exhibits problems of logical dependence. More precisely, the following proposition shows that one of the constraints (iv) and (v) is redundant.

**Proposition 1.** *$d$ satisfies (i), (ii), and (iv) if and only if it satisfies (i), (ii), and (v).*

**Proof**. Suppose that $d$ satisfies (i), (ii), and (iv). Consider a triple $x, y, z$. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and that

$$d_S(z, x) > d_S(y, x),$$

for some $S \in \mathcal{E}$. Then, $d_S(z, x) = 1$, $d_S(y, x) = 0$. But then, $d_{S^c}(x, z) = 0$, $d_{S^c}(x, y) = 1$. This implies that

$$d_{S^c}(x, y) > d_{S^c}(x, z),$$

contradicting (iv). Therefore, $d$ satisfies (i), (ii), and (v). Suppose that $d$ satisfies (i), (ii), and (v). Consider a triple $x, y, z$. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and that

$$d_S(x, y) > d_S(x, z),$$

for some $S \in \mathcal{E}$. Then, $d_S(x, y) = 1$, $d_S(x, z) = 0$. But then, $d_{S^c}(y, x) = 0$, $d_{S^c}(z, x) = 1$. This implies that

$$d_{S^c}(z, x) > d_{S^c}(y, x),$$

contradicting (v). Therefore, $d$ satisfies (i), (ii), and (iv). ∎

Moreover, the following proposition shows that one of the constraints (vi) and (vii) is redundant.

**Proposition 2.** *$d$ satisfies (i), (ii), and (vi) if and only if it satisfies (i), (ii), and (vii).*

**Proof**. Suppose that $d$ satisfies (i), (ii), and (vi). Consider a triple $x, y, z$. Suppose that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and that

$$d_S(z,y) + d_S(y,x) < d_S(z,x),$$

for some $S \in \mathcal{E}$. Thus, $d_S(z,y) = 0$, $d_S(y,x) = 0$, and $d_S(z,x) = 1$. But then, $d_{S^c}(y,z) = 1$, $d_{S^c}(x,y) = 1$, and $d_{S^c}(x,z) = 0$. This implies that

$$d_{S^c}(x,y) + d_{S^c}(y,z) > 1 + d_{S^c}(x,z),$$

contradicting (vi). Therefore, $d$ satisfies (i), (ii), and (vii). Suppose that $d$ satisfies (i), (ii), and (vii). Consider a triple $x, y, z$. Suppose that there exists $\mathbf{p} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$, and that

$$d_S(x,y) + d_S(y,z) > 1 + d_S(x,z),$$

for some $S \in \mathcal{E}$. Then, $d_S(x,y) = 1$, $d_S(y,z) = 1$, and $d_S(x,z) = 0$. But then, $d_{S^c}(y,x) = 0$, $d_{S^c}(z,y) = 0$, and $d_{S^c}(z,x) = 1$. This implies that

$$d_{S^c}(z,y) + d_{S^c}(y,x) < d_{S^c}(z,x),$$

contradicting (vii). Therefore, $d$ satisfies (i), (ii), and (vi). ■

We will use Propositions 1 and 2 in the next section. There, we will provide a natural generalization of Sethuraman et al.'s approach, specifying two integer programs in which variables $d_S(x,y)$ are allowed to assume values in the set $\{0, \frac{1}{2}, 1\}$.

# 4 Ternary integer programming and Arrovian social welfare functions with ties: a correspondence theorem

In this section, we first introduce a generalization of Sethuraman et al.'s IP0 to the case where $d_S(x,y) = \frac{1}{2}$, for some $(x,y) \in NTR$ and for some $S \in \mathcal{E}$. We will show that this ternary program on $\Omega$ - which we will call IP1 - can be used to represent ASWFs with ties. IP1 consists of the following set of constraints:

$$d_E(x,y) = 1, \tag{1}$$

for all $(x, y) \in NTR$;

$$d_S(x, y) + d_{S^c}(y, x) = 1, \tag{2}$$

for all $(x, y) \in NTR$ and for all $S \in \mathcal{E}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{3}$$

if $d_{A \cup U \cup V}(x, y), d_{B \cup U \cup W}(y, z), d_{C \cup V \cup W}(z, x) \in \{0, 1\}$;

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) = \frac{3}{2}, \tag{4}$$

if $d_{A \cup U \cup V}(x, y) = \frac{1}{2}$ or $d_{B \cup U \cup W}(y, z) = \frac{1}{2}$ or $d_{C \cup V \cup W}(z, x) = \frac{1}{2}$, for all triples of alternatives $x, y, z$ and for all disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$.

In fact, we propose now a result which establishes a one-to-one correspondence between the set of the solutions to IP1 on a given $\Omega$ and the set of the ASWFs with ties on the same $\Omega$.

**Theorem 1.** *Consider a domain $\Omega$. Given an ASWF on $\Omega$, $f$, there exists a unique solution to IP1 on $\Omega$, $d$, which corresponds to $f$. Given a solution to IP1 on $\Omega$, $d$, there exists a unique ASWF on $\Omega$, $f$, which corresponds to $d$.*

**Proof**. Consider a domain $\Omega$ and an ASWF on $\Omega$, $f$. Determine $d$ as follows. Given $(x, y) \in NTR$ and $S \in \mathcal{E}$, consider $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$. Let $d_S(x, y) = 1$ if $xP(f(\mathbf{P}))y$, $d_S(x, y) = \frac{1}{2}$ if $xI(f(\mathbf{P}))y$, $d_S(x, y) = 0$ if $yP(f(\mathbf{P}))x$. Then, for each $(x, y) \in NTR$ and for each $S \in \mathcal{E}$, we have $xP(f(\mathbf{P}))y$ if and only if $d_S(x, y) = 1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x, y) = \frac{1}{2}$, $yP(f(\mathbf{P}))x$ if and only if $d_S(x, y) = 0$, for all $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$, as $f$ satisfies IIA. $d$ satisfies (1), as $f(\mathbf{P})$ satisfies PO, and (2), as $f(\mathbf{P})$ is a complete binary relation on $\mathcal{A}$, for all $\mathbf{P} \in \Omega^n$. Consider a triple $x, y, z$, and disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$. Moreover, consider $\mathbf{P} \in \Omega^n$. Then, by Conditions $(*)$, we have: $x\mathbf{p}_i y$, for all $i \in A \cup U \cup V$; $y\mathbf{p}_i x$, for all $i \in (A \cup U \cup V)^c$; $y\mathbf{p}_i z$, for all $i \in B \cup U \cup W$; $z\mathbf{p}_i y$, for all $i \in (B \cup U \cup W)^c$; $z\mathbf{p}_i x$, for all $i \in C \cup V \cup W$; $x\mathbf{p}_i z$, for all $i \in (C \cup V \cup W)^c$. Suppose that $d_{A \cup U \cup V}(x, y), d_{B \cup U \cup W}(y, z), d_{C \cup V \cup W}(z, x) \in \{0, 1\}$ and

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) > 2.$$

Then, we have $xP(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$, a contradiction. Suppose that $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) < \frac{3}{2}.$$

Consider the following three cases. First, $d_{B\cup U\cup W}(y,z) = 0$ and $d_{C\cup V\cup W}(z,x) = 0$. Then, we have $zP(f(\mathbf{P}))yI(f(\mathbf{P}))x$ and $xP(f(\mathbf{P}))z$, a contradiction. Second, $d_{B\cup U\cup W}(y,z) = \frac{1}{2}$ and $d_{C\cup V\cup W}(z,x) = 0$. Then, we have $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $xP(f(\mathbf{P}))z$, a contradiction. Third, $d_{B\cup U\cup W}(y,z) = 0$ and $d_{C\cup V\cup W}(z,x) = \frac{1}{2}$. Then, we have $zI(f(\mathbf{P}))xI(f(\mathbf{P}))y$ and $zP(f(\mathbf{P}))y$, a contradiction. Suppose now that $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > \frac{3}{2}.$$

Consider the following three cases. First, $d_{B\cup U\cup W}(y,z) = 1$ and $d_{C\cup V\cup W}(z,x) = 1$. Then, we have $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$, a contradiction. Second, $d_{B\cup U\cup W}(y,z) = \frac{1}{2}$ and $d_{C\cup V\cup W}(z,x) = 1$. Then, we have $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$, a contradiction. Third, $d_{B\cup U\cup W}(y,z) = 1$ and $d_{C\cup V\cup W}(z,x) = \frac{1}{2}$. Then, we have $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zI(f(\mathbf{P}))x$, a contradiction. Therefore, $d$ satisfies (3) and (4). Hence, $d$ is a solution to IP1 on $\Omega$ which corresponds to $f$. Suppose that $d$ is not unique. Then, there exist a solution to IP1 on $\Omega$, $d'$, $(x,y) \in NTR$, and $S \in \mathcal{E}$ such that $d_S(x,y) \neq d'_S(x,y)$. Consider $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$. Then, we have $xP(f(\mathbf{P}))y$ and $xI(f(\mathbf{P}))y$, or, $yP(f(\mathbf{P}))x$ and $xI(f(\mathbf{P}))y$, or, $xP(f(\mathbf{P}))y$ and $yP(f(\mathbf{P}))x$, a contradiction. But then, $d$ is unique. Now consider a solution to IP1 on $\Omega$, $d$. Determine $f$ as follows. Given $(x,y) \in TR$, let $xP(f(\mathbf{P}))y$, for all $\mathbf{P} \in \Omega^n$. Given $(x,y) \in NTR$ and $\mathbf{P} \in \Omega^n$, let $S \in \mathcal{E}$ be the set of agents such that $x\mathbf{p}_i y$, for all $i \in S$, and $y\mathbf{p}_i x$, for all $i \in S^c$. Let $xP(f(\mathbf{P}))y$ if $d_S(x,y) = 1$, $xI(f(\mathbf{P}))y$ if $d_S(x,y) = \frac{1}{2}$, and $yP(f(\mathbf{P}))x$ if $d_S(x,y) = 0$. $f(\mathbf{P})$ is a complete binary relation on $\mathcal{A}$, for all $\mathbf{P} \in \Omega^n$, by construction and by (2). Now, we show that $f(\mathbf{P})$ is also a transitive binary relation on $\mathcal{A}$, for all $\mathbf{P} \in \Omega^n$. Consider a triple $x, y, z$ and a preference profile $\mathbf{P} \in \Omega^n$. Then, there exist three nonempty sets $H$, $I$, $J$ such that $x\mathbf{p}_i y$, for all $i \in H$, $y\mathbf{p}_i x$, for all $i \in H^c$, $y\mathbf{p}_i z$, for all $i \in I$, $z\mathbf{p}_i y$, for all $i \in I^c$, $z\mathbf{p}_i x$, for all $i \in J$, $x\mathbf{p}_i z$, for all $i \in J^c$. Let $A = H \setminus (I \cup J)$, $B = I \setminus (H \cup J)$, $C = J \setminus (H \cup I)$, $U = H \cap I$, $V = H \cap J$, $W = I \cap J$. Then, $A, B, C, U, V, W \in \mathcal{E}$ are disjoint sets of agents whose union includes all agents and which satisfy Conditions ($*$). Moreover, they satisfy $A \cup U \cup V = H$, $B \cup U \cup W = I$, $C \cup V \cup W = J$. Consider the following eight cases. First, $xP(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{A\cup U\cup V}(x,y) = 1$, $d_{B\cup U\cup W}(y,z) = 1$, $d_{C\cup V\cup W)}(z,x) = 1$, and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > 2,$$

contradicting (3). Second, $xP(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $xI(f(\mathbf{P}))z$. Then, $d_{C\cup V\cup W)}(z,x) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Third, $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Fourth, $xI(f(\mathbf{P}))yP(f(\mathbf{P}))z$ and $xI(f(\mathbf{P}))z$. Then, $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Fifth, $xP(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{B\cup U\cup W}(y,z) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Sixth, $xP(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $xI(f(\mathbf{P}))z$. Then, $d_{B\cup U\cup W}(y,z) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > \frac{3}{2},$$

contradicting (4). Seventh, $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $xP(f(\mathbf{P}))z$. Then, $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) < \frac{3}{2},$$

contradicting (4). Eighth, $xI(f(\mathbf{P}))yI(f(\mathbf{P}))z$ and $zP(f(\mathbf{P}))x$. Then, $d_{A\cup U\cup V}(x,y)=\frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y)+d_{B\cup U\cup W}(y,z)+d_{C\cup V\cup W}(z,x)>\frac{3}{2},$$

contradicting (4). $f$ satisfies PO as, for all $(x,y)\in TR$, we have $xp(f(\mathbf{P}))y$, for all $\mathbf{P}\in\Omega^n$; moreover, for all $(x,y)\in NTR$ and for all $\mathbf{P}\in\Omega^n$, $x\mathbf{p}_iy$, for all $i\in E$, implies $xP(f(\mathbf{P}))y$, by (1). $f$ satisfies IIA as, for each $(x,y)\in NTR$ and for each $S\in\mathcal{E}$, we have $xP(f(\mathbf{P}))y$ if and only if $d_S(x,y)=1$, $xI(f(\mathbf{P}))y$ if and only if $d_S(x,y)=\frac{1}{2}$, and $yP(f(\mathbf{P}))x$ if and only if $d_S(x,y)=0$, for all $\mathbf{P}\in\Omega^n$ such that $x\mathbf{p}_iy$, for all $i\in S$, and $y\mathbf{p}_ix$, for all $i\in S^c$. Hence, $f$ is an ASWF on $\Omega$, which corresponds to $d$. Suppose that $f$ is not unique. Then, there exist an ASWF on $\Omega$, $f'$, $(x,y)\in NTR$ and $\mathbf{P}\in\Omega^n$ such that we have $xf(\mathbf{P})y$ but not $xf'(\mathbf{P})y$. Let $S\in\mathcal{E}$ be the set such that $x\mathbf{p}_iy$, for all $i\in S$, and $y\mathbf{p}_ix$, for all $i\in S^c$. Then, $d_S(x,y)=1$ and $d_S(x,y)=0$, or, $d_S(x,y)=\frac{1}{2}$ and $d_S(x,y)=0$, a contradiction. But then, $f$ is unique. ■

We introduce now a second ternary IP on $\Omega$, which incorporates - like Sethuraman et al.'s IP0$'$ - a reformulation of Kalai and Muller's Conditions I-IV. In constructing it, we draw the consequences of Propositions 1 and 2 and eliminate the redundancies inherent in Sethuraman et al.'s IP0$'$. In fact, this second ternary IP - which we will call IP1$'$ - consists of constraints (1), (2), and the following four logically independent constraints:

$$d_S(x,y)\le d_S(x,z), \tag{5}$$

if $d_S(x,y)\in\{0,1\}$;

$$d_S(x,y)<d_S(x,z), \tag{6}$$

if $d_s(x,y)=\frac{1}{2}$, for all triples $x,y,z$ such that there exist $\mathbf{p},\mathbf{q}\in\Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, and for all $S\in\mathcal{E}$;

$$d_S(x,y)+d_S(y,z)\le 1+d_S(x,z), \tag{7}$$

if $d_s(x,y),d_s(y,z)\in\{0,1\}$;

$$d_S(x,y)+d_S(y,z)=\frac{1}{2}+d_S(x,z), \tag{8}$$

if $d_S(x,y)=\frac{1}{2}$ or $d_S(y,z)=\frac{1}{2}$, for all triples $x,y,z$ such that there exist $\mathbf{p},\mathbf{q}\in\Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, and for all $S\in\mathcal{E}$. [3]

[3]We notice that, in our formulation of (7) and (8), we suppose that there exist $\mathbf{p},\mathbf{q}\in\Omega$ which satisfy $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$, whereas Sethuraman et al. (2003), in their formulation of (vii) and (viii), supposed that there exists only $\mathbf{p}\in\Omega$ which satisfies $x\mathbf{p}y\mathbf{p}z$.

In the remainder of this section, we prove two propositions which establish the relationships between IP1 and IP1$'$.

**Proposition 3.** *If $d$ is a solution to IP1 on $\Omega$, then it is a solution to IP$1'$ on the same $\Omega$.*

**Proof**. Let $d$ be a solution to IP1 on $\Omega$. Consider a triple $x, y, z$ and $S \in \mathcal{E}$. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ which satisfy $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Let $U = S$, $W = S^c$, and $A = B = C = V = \emptyset$. Then, $A, B, C, U, V, W$ are sets whose union includes all agents and which satisfy Conditions $(*)$. Suppose that $d_S(x, y) \in \{0, 1\}$ and $d_S(x, y) > d_S(x, z)$. Consider the following two cases. First, $d_S(x, z) \in \{0, 1\}$. Then,

$$d_U(x, y) + d_{U \cup W}(y, z) + d_W(z, x) > 2,$$

contradicting (3). Second, $d_S(x, z) = \frac{1}{2}$. Then,

$$d_U(x, y) + d_{U \cup W}(y, z) + d_W(z, x) > \frac{3}{2},$$

contradicting (4). Therefore, $d$ satisfies (5). Suppose now that $d_S(x, y) = \frac{1}{2}$ and $d_S(x, y) \geq d_S(x, z)$. Then,

$$d_U(x, y) + d_{U \cup W}(y, z) + d_W(z, x) > \frac{3}{2},$$

contradicting (4). Therefore, $d$ satisfies (6). Consider a triple $x, y, z$ and $S \in \mathcal{E}$. Suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Let $C = S^c$, $U = S$, and $A = B = V = W = \emptyset$. Then, $A, B, C, U, V, W$ are sets whose union includes all agents and which satisfy Conditions $(*)$. Suppose that $d_S(x, y), d_S(y, z) \in \{0, 1\}$ and $d_S(x, y) + d_S(y, z) > 1 + d_S(x, z)$. Consider the following two cases. First, $d_S(x, z) \in \{0, 1\}$. Then,

$$d_U(x, y) + d_U(y, z) + d_C(z, x) > 2,$$

contradicting (3). Second, $d_S(x, z) = \frac{1}{2}$. Then,

$$d_U(x, y) + d_U(y, z) + d_C(z, x) > \frac{3}{2},$$

contradicting (4). Therefore, $d$ satisfies (7). Suppose now that $d_S(x, y) = \frac{1}{2}$ and $d_S(x, y) + d_S(y, z) < \frac{1}{2} + d_S(x, z)$. Then,

$$d_U(x, y) + d_U(y, z) + d_C(z, x) < \frac{3}{2},$$

contradicting (4). Suppose that $d_S(x,y) = \frac{1}{2}$ and $d_S(x,y) + d_S(y,z) > \frac{1}{2} + d_S(x,z)$. Then,

$$d_U(x,y) + d_U(y,z) + d_C(z,x) > \frac{3}{2},$$

contradicting (4). Therefore, $d$ satisfies (8). Hence, $d$ is a solution to IP1′ on $\Omega$. ■

The following result shows that the converse of Proposition 3 holds - and IP1 and IP1′ coincide - when $n = 2$.

**Proposition 4.** *Let $n = 2$. If $d$ is a solution to IP1′ on $\Omega$, then it is a solution to IP1 on the same $\Omega$.*

**Proof**. Let $n = 2$. Let $d$ be a solution to IP1′ on $\Omega$. Consider a triple $x, y, z$ and disjoint and possibly empty sets $A, B, C, U, V, W \in \mathcal{E}$ whose union includes all agents and which satisfy Conditions $(*)$. Suppose that $d_{A\cup U\cup V}(x,y), d_{B\cup U\cup W}(y,z), d_{C\cup V\cup W}(z,x) \in \{0,1\}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) > 2.$$

Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Then,

$$d_{\{2\}}(y,z) + d_{\{2\}}(z,x) > 1 + d_{\{2\}}(y,x),$$

contradicting (7). The cases where $B \neq \emptyset$, $V \neq \emptyset$, and $C \neq \emptyset$, $U \neq \emptyset$ lead, *mutatis mutandis*, to the same contradiction. Consider the case where $U \neq \emptyset$ and $V \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}x\mathbf{q}y$. Suppose that $U = \{1\}$ and $V = \{2\}$. Then,

$$d_{\{2\}}(z,x) > d_{\{2\}}(z,y),$$

contradicting (5). The cases where $V \neq \emptyset$, $W \neq \emptyset$, and $U \neq \emptyset$, $W \neq \emptyset$, lead, *mutatis mutandis*, to the same contradiction. Therefore, $d$ satisfies (3). Suppose that $d_{A\cup U\cup V}(x,y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x,y) + d_{B\cup U\cup W}(y,z) + d_{C\cup V\cup W}(z,x) < \frac{3}{2}.$$

Consider the case where $A \neq \emptyset$ and $B \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}x\mathbf{q}z$. Suppose that $A = \{1\}$ and $B = \{2\}$. Then, $d_{\{2\}}(y,x) = \frac{1}{2}$ and

$$d_{\{2\}}(y,x) \geq d_{\{2\}}(y,z),$$

contradicting (6). The case where $A \neq \emptyset$ and $C \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Suppose that $d_{\{2\}}(y, z) = 0$ and $d_{\{2\}}(z, x) = 0$. Then,

$$d_{\{1\}}(x, z) + d_{\{1\}}(z, y) > 1 + d_{\{1\}}(x, y),$$

contradicting (7). Suppose that $d_{\{2\}}(y, z) = \frac{1}{2}$ and $d_{\{2\}}(z, x) = 0$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) < \frac{1}{2} + d_{\{2\}}(y, x),$$

contradicting (8). Consider the case where $U \neq \emptyset$ and $C \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Suppose that $U = \{1\}$ and $C = \{2\}$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) < \frac{1}{2} + d_{\{1\}}(x, z),$$

contradicting (8). The case where $V \neq \emptyset$ and $B \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Suppose that $d_{A\cup U\cup V}(x, y) = \frac{1}{2}$ and

$$d_{A\cup U\cup V}(x, y) + d_{B\cup U\cup W}(y, z) + d_{C\cup V\cup W}(z, x) > \frac{3}{2}.$$

Consider the case where $A \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}z\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $A = \{1\}$ and $W = \{2\}$. Suppose that $d_{\{2\}}(y, z) = 1$ and $d_{\{2\}}(z, x) = 1$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) > 1 + d_{\{2\}}(y, x),$$

a contradicting (7). Suppose that $d_{\{2\}}(y, z) = \frac{1}{2}$ and $d_{\{2\}}(z, x) = 1$. Then,

$$d_{\{2\}}(y, z) + d_{\{2\}}(z, x) > \frac{1}{2} + d_{\{2\}}(y, x),$$

contradicting (8). Consider the case where $U \neq \emptyset$ and $C \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Suppose that $U = \{1\}$ and $C = \{2\}$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > \frac{1}{2} + d_{\{1\}}(x, z),$$

contradicting (8). The case where $V \neq \emptyset$ and $B \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Consider the case where $U \neq \emptyset$ and $W \neq \emptyset$. Then, there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Suppose that $U = \{1\}$ and $W = \{2\}$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) \geq d_{\{1\}}(x, z),$$

contradicting (6). The case where $V \neq \emptyset$ and $W \neq \emptyset$ leads, *mutatis mutandis*, to the same contradiction. Therefore, $d$ satisfies (4). Hence, $d$ is a solution to IP1 on $\Omega$. ■

# 5 Integer programming and nondictatorial Arrovian social welfare functions without ties: a new proof of Kalai and Muller's Theorem 2

In this section and the next, we use the integer programs developed above to deal with the issues concerning the dictatorship property of ASWFs. To begin with, we focus here on ASWFs without ties.

Kalai and Muller (1977) were the first who provided a complete characterization of the domains of antisymmetric preference orderings which admit nondictatorial ASWFs without ties. They did this by means of two theorems. In their Theorem 1, they showed that, for a given domain $\Omega$, there exists a nondictatorial ASWF without ties for $n > 2$ if and only if, for the same $\Omega$, there exists a nondictatorial ASWF without ties for $n = 2$. In their Theorem 2, Kalai and Muller showed that there exists a nondictatorial ASWF without ties on $\Omega$ for $n \geq 2$ if and only if $\Omega$ satisfies the conditions of KM decomposability introduced in Section 2.

Sethuraman et al. opened the way to an analysis of the problem of dictatorship in terms of integer programming. More precisely, in the 2003 paper, they showed a result establishing a be-univocal relation between the solutions of IP0 for $n = 2$ and its solutions for $n > 2$. Since their arguments can straightforwardly be re-expressed in terms of IP1, their result can be stated as follows.

**Theorem 2.** *There exists a nondictatorial binary solution to IP1 on $\Omega$, $d$, for $n = 2$, if and only if there exists a nondictatorial binary solution to IP1 on $\Omega$, $d^*$, for $n > 2$.*

Kalai and Muller's Theorem 1 can therefore be obtained, by our Theorem 1, as a corollary of Theorem 2.

**Corollary 1.** *There exists a nondictatorial ASWF without ties on* $\Omega$, $f$, *for* $n = 2$, *if and only if there exists a nondictatorial ASWF without ties on* $\Omega$, $f^*$, *for* $n > 2$.

Now, we go forward along the line opened by Sethuraman et al. (2003), providing a characterization of domains admitting nondictatorial binary solutions to IP1. As it will be made clear shortly, this result is the heart of the new, simpler proof of Kalai and Muller's Theorem 2 for ASWFs without ties, in terms of integer programming.

In order to obtain our characterization theorem, we need to introduce a reformulation of Kalai and Muller's concept of decomposability suitable to be applied within the analytical context of IP1. We will show below that this reformulation is equivalent to the original version proposed by Kalai and Muller. Our reformulation is based on the existence of two sets, $R_1, R_2 \in \mathcal{A}^2$ - instead of only one - which satisfy the two conditions we are going to introduce.

Consider a set $R \subset \mathcal{A}^2$. Consider the following conditions on $R$.

*Condition 1. For all triples* $x, y, z$, *if there exist* $\mathbf{p}, \mathbf{q} \in \Omega$ *satisfying* $x\mathbf{p}y\mathbf{p}z$ *and* $y\mathbf{q}z\mathbf{q}x$, *then* $(x, y) \in R$ *implies that* $(x, z) \in R$.

*Condition 2. For all triples* $x, y, z$, *if there exist* $\mathbf{p}, \mathbf{q} \in \Omega$ *satisfying* $x\mathbf{p}y\mathbf{p}z$ *and* $z\mathbf{q}y\mathbf{q}x$, *then* $(x, y) \in R$ *and* $(y, z) \in R$ *imply that* $(x, z) \in R$.

A domain $\Omega$ is said to be decomposable if and only if there exist two sets $R_1$ and $R_2$, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, $R_i$ satisfies Conditions 1 and 2.

With regard to this definition of a decomposable domain, let us notice the main differences with Kalai and Muller's original notion, introduced to make it compatible with the integer programming analytical setting: Conditions 1 and 2 differ from Conditions I and II as the former refer to triples, rather than pairs, of alternatives. Moreover, Condition 2 is reformulated in terms of a pair of preference orderings - instead of only one - consistently with our formulation of constraints (7) and (8). Also, our formulation does not require that $R_1$ and $R_2$ contain $TR$, whereas Kalai and Muller's one requires that $R$ contains $TR$. In particular, let us stress that our definition requires that $R_1$ and $R_2$ satisfy only two conditions - instead of four, as in Kalai and Muller's version. As Corollary 3 below makes it clear, this implies a redundancy of Kalai and Muller's Conditions II and IV, which parallels the redundancy of constraints (v) and (vii) proved in Propositions 2 and 3.

On the basis of our reformulation of the concept of decomposability, we state and prove now the characterization theorem.

**Theorem 3.** *There exists a nondictatorial binary solution to IP*$1'$ *on* $\Omega$, $d$, *for* $n = 2$, *if and only if* $\Omega$ *is decomposable.*

**Proof**. Let $d$ be a nondictatorial binary solution to IP$1'$ on $\Omega$, for $n = 2$. Let $R_1 = \{(x, y) \in NTR : d_{\{1\}}(x, y) = 1\}$ and $R_2 = \{(x, y) \in NTR : d_{\{2\}}(x, y) = 1\}$. Then, for all $(x, y) \in NTR$, $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$, as $d$ satisfies (2). Moreover, $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, as $d$ is nondictatorial. Consider a triple $x, y, z$ and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $(x, y) \in R_1$ and $(x, z) \notin R_1$ Then, $d_{\{1\}}(x, y) = 1$ and

$$d_{\{1\}}(x, y) > d_{\{1\}}(x, z),$$

contradicting (5). Hence, $R_i$, $i = 1, 2$, satisfies Condition 1. Consider a triple $x, y, z$ and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x, y), (y, z) \in R_1$, and $(x, z) \notin R_1$. Then, $d_{\{1\}}(x, y) = 1$, $d_{\{1\}}(y, z) = 1$, and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z),$$

contradicting (7). Hence, $R_i$, $i = 1, 2$, satisfies Condition 2. We have proved that $\Omega$ is decomposable. Conversely, suppose that $\Omega$ is decomposable. Then, there exist two sets $R_1$ and $R_2$, with $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$; moreover, $R_i$ satisfies Conditions 1 and 2. Determine $d$ as follows. For each $(x, y) \in NTR$, let $d_\emptyset(x, y) = 0$, $d_E(x, y) = 1$; moreover, let $d_{\{i\}}(x, y) = 1$ if and only if $(x, y) \in R_i$; $d_{\{i\}}(x, y) = 0$ if and only if $(x, y) \notin R_i$, for $i = 1, 2$. Then, $d$ satisfies (1) and (2) as, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(y, x) \notin R_2$. Consider a triple $x, y, z$ and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x, y) > d_{\{1\}}(x, z).$$

Then, we have $(x, y) \in R_1$ and $(x, z) \notin R_1$, contradicting Condition 1. Therefore, $d$ satisfies (5). Consider a triple $x, y, z$ and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > 1 + d_{\{1\}}(x, z).$$

Then, we have $(x, y), (y, z) \in R_1$ and $(x, z) \notin R_1$, contradicting Condition 2. Therefore, $d$ satisfies (7). $d$ is nondictatorial as $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1, 2$. Hence, $d$ is a nondictatorial binary solution to IP$1'$ on $\Omega$. ■

The previous result provides a simplified proof of Kalai and Muller's Theorem 2 since this theorem can be immediately obtained as a corollary of Theorem 3.

**Corollary 2.** *There exists a nondictatorial ASWF without ties on $\Omega$, $f$, for $n \geq 2$, if and only if $\Omega$ is decomposable.*

**Proof**. It is a straightforward consequence of Propositions 3 and 4, Theorems 1 and 3, and Corollary 1. ■

From Theorem 3, we obtain a further corollary, which - as anticipated above - establishes the equivalence between our notion of decomposability and Kalai and Muller's notion, and implies that Kalai and Muller's Conditions II and IV are redundant.

**Corollary 3.** *$\Omega$ is KM decomposable if and only if it s decomposable.*

**Proof**. It is an immediate consequence of Kalai and Muller's Theorem 2 and of Corollary 2. ■

# 6 Integer programming and nondictatorial Arrovian social welfare functions with ties: a new characterization theorem

In the analysis developed above, the integer programming setup - in particular the IP1 introduced in Section 4 - has proved to be an effective tool in order to provide simplified demonstrations of Kalai and Muller's crucial results on ASWFs without ties.

In this section, we further exploit IP1 to progress in the investigation of nondictatorship. As already reminded, Arrow's impossibility theorem is established for ASWFs admitting ties in their range and defined on the unrestricted domain of preference orderings. Kalai and Muller's characterization theorem overcomes Arrow's impossibility result by considering ASWFs without ties in their range, defined on the domain of antisymmetric preference orderings.

We take a step forward along this way: our main theorem establishes a characterization of the domains of antisymmetric preference orderings admitting nondictatorial ASWFs with ties.

We start our analysis by proving the following result, which extends Theorem 2 above to the case of ternary solutions to IP1.

**Theorem 4.** *There exists a nondictatorial ternary solution to IP1 on* $\Omega$*,* $d$*, for* $n = 2$*, if and only if there exists a nondictatorial ternary solution to IP1 on* $\Omega$*,* $d^*$*, for* $n > 2$*.*

**Proof**. Let $d$ be a nondictatorial ternary solution to IP1 on $\Omega$ for $n = 2$. Determine $d^*$ as follows. Given $(x, y) \in NTR$ and $S \in \mathcal{E}$, let $d^*_S(x, y) = 1$ if $1, 2 \in S$; $d_S(x, y) = 0$ if $1, 2 \in S^c$; $d^*_S(x, y) = d_{\{1\}}(x, y)$ and $d^*_{S^c}(y, x) = d_{\{2\}}(y, x)$ if $1 \in S$ and $2 \in S^c$. Then, it is straightforward to verify that $d^*$ satisfies (1)-(4) and that is nondictatorial. Hence, $d^*$ is a nondictatorial ternary solution to IP1 on $\Omega$, for $n > 2$. Conversely, let $d^*$ be a nondictatorial ternary solution to IP1 on $\Omega$ for $n > 2$. Determine $d$ as follows. Consider $(u, v) \in NTR$ and $\bar{S} \in \mathcal{E}$ such that $d^*_{\bar{S}}(u, v) = \frac{1}{2}$. Given $(x, y) \in NTR$, let $d_{\{1,2\}}(x, y) = 1$, $d_\emptyset(x, y) = 0$, $d_{\{1\}}(x, y) = d^*_{\bar{S}}(x, y)$, $d_{\{2\}}(x, y) = d^*_{\bar{S}}(x, y)$. Then, it is straightforward to verify that $d$ satisfies (1) and (2). Moreover, by Proposition 3, $d$ satisfies (5)-(8) as $d^*$ is a solution to IP1 on $\Omega$. But then, $d$ is a solution to IP1$'$ on $\Omega$ and this, in turn, implies that it is a solution to IP1 on $\Omega$, by Proposition 4. Finally, $d$ is nondictatorial as $d_{\{1\}}(u, v) = \frac{1}{2}$. Hence, $d$ is a nondictatorial ternary solution to IP1 on $\Omega$, for $n = 2$ ■

From Theorem 4, we obtain the following corollary, which extends Kalai and Muller's Theorem 1 to the case of ASWFs with ties. It is an immediate consequence of our Theorem 1 in Section 4.

**Corollary 4.** *There exists a nondictatorial ASWF with ties on* $\Omega$*,* $f$*, for* $n = 2$*, if and only if there exists a nondictatorial ASWF with ties on* $\Omega$*,* $f^*$*, for* $n > 2$*.*

In order to obtain our characterization theorem for ASWFs with ties, we need to introduce a new notion of decomposability, stricter than the one introduced in Section 5 (which was shown to be equivalent to the notion of KM decomposability). We define it as "strict decomposability." The next section will be devoted to establish the exact relationship between the two notions of decomposability and strict decomposability.

Then, consider a set $R \subset \mathcal{A}^2$. Consider the following conditions on $R$.

*Condition 3. There exists a set* $R^* \subset \mathcal{A}^2$*, with* $R \cap R^* = \emptyset$*, such that, for all triples* $x, y, z$*, if there exist* $\mathbf{p}, \mathbf{q} \in \Omega$ *satisfying* $x\mathbf{p}y\mathbf{p}z$ *and* $y\mathbf{q}z\mathbf{q}x$*, then* $(x, y) \in R^*$ *implies that* $(x, z) \in R$*.*

*Condition 4. There exists a set* $R^* \subset \mathcal{A}$*, with* $R \cap R^* = \emptyset$*, such that, for all triples of alternatives* $x, y, z$*, if there exist* $\mathbf{p}, \mathbf{q} \in \Omega$ *satisfying* $x\mathbf{p}y\mathbf{p}z$

*and* $z\mathbf{q}y\mathbf{q}x$*, then* $(x,y) \in R$ *and* $(y,z) \in R^*$ *imply that* $(x,z) \in R$*, and* $(x,y) \in R^*$ *and* $(y,z) \in R$ *imply that* $(x,z) \in R$*.*

A domain $\Omega$ is said to be strictly decomposable if and only if there exist four sets $R_1$, $R_2$, $R_1^*$, and $R_2^*$, with $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1,2$, such that, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(x,y) \notin R_1^*$ and $(y,x) \notin R_2$; $(x,y) \in R_1^*$ if and only if $(y,x) \in R_2^*$; moreover, $R_i$, $i = 1,2$, satisfies Condition 1; $R_i$ and $R_i^*$, $i = 1,2$, satisfy Condition 2; each pair $(R_i, R_i^*)$, $i = 1,2$, satisfies Conditions 3 and 4.

On the basis of the notion of strict decomposability, we provide now the characterization of domains admitting nondictatorial ternary solutions to IP1.

**Theorem 5.** *There exists a nondictatorial ternary solution to IP*$1'$ *on* $\Omega$*,* $d$*, for* $n = 2$*, if and only if* $\Omega$ *is strictly decomposable.*

**Proof**. Let $d$ be a nondictatorial ternary solution to IP$1'$ on $\Omega$, for $n = 2$. Let $R_1 = \{(x,y) \in NTR : d_{\{1\}}(x,y) = 1\}$, $R_2 = \{(x,y) \in NTR : d_{\{2\}}(x,y) = 1\}$, $R_1^* = \{(x,y) \in NTR : d_{\{1\}}(x,y) = \frac{1}{2}\}$, $R_2^* = \{(x,y) \in NTR : d_{\{2\}}(x,y) = \frac{1}{2}\}$. Consider $(x,y) \in NTR$. Suppose that $(x,y) \in R_1$ and $(x,y) \in R_1^*$. Then, $d_{\{1\}}(x,y) = 1$ and $d_{\{1\}}(x,y) = \frac{1}{2}$, a contradiction. Suppose that $(x,y) \in R_1$ and $(y,x) \in R_2$. Then, $d_{\{1\}}(x,y) = 1$ and $d_{\{2\}}(y,x) = 1$, contradicting (2). Suppose that $(x,y) \notin R_1^*$ and $(y,x) \notin R_2$ and $(x,y) \notin R_1$. Then, $d_{\{1\}}(x,y) \neq \frac{1}{2}$, $d_{\{1\}}(x,y) \neq 0$, and $d_{\{1\}}(x,y) \neq 1$, a contradiction. Suppose that $(x,y) \in R_1^*$ and $(y,x) \notin R_2^*$. Then, $d_{\{1\}}(x,y) = \frac{1}{2}$ and $d_{\{2\}}(y,x) \neq \frac{1}{2}$, contradicting (2). Hence, for all $(x,y) \in NTR$, $(x,y) \in R_1$ if and only if $(x,y) \notin R_1^*$ and $(y,x) \notin R_2$; $(x,y) \in R_1^*$ if and only if $(y,x) \in R_2^*$. Suppose that $R_1 = NTR$. Then, $d$ is dictatorial, a contradiction. Hence, $R_i \subsetneqq NTR$, $i = 1,2$. Suppose that $R_i^* = \emptyset$, $i = 1,2$. Then, $d$ is a binary solution, a contradiction. Hence, $\emptyset \subsetneqq R_i^* \subset NTR$. Moreover, by using the same argument developed in the proof of Theorem 3, it can be shown that $R_i$, $i = 1,2$, satisfies Conditions 1 and 2. Consider a triple $x,y,z$ and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x,y) \in R_1^*$, $(y,z) \in R_1^*$, and $(x,z) \notin R_1^*$. Then, $d_{\{1\}}(x,y) = \frac{1}{2}$, $d_{\{1\}}(y,z) = \frac{1}{2}$, and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) \neq \frac{1}{2} + d_{\{1\}}(x,z),$$

contradicting (8). Hence, $R_i^*$ satisfies Condition 2, $i = 1,2$. Consider a triple $x,y,z$ and suppose that there exist $\mathbf{p},\mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$.

Moreover, suppose that $(x, y) \in R_1^*$ and $(x, z) \notin R_1$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) \geq d_{\{1\}}(x, z),$$

contradicting (6). Hence, each pair $(R_i, R_i^*)$, $i = 1, 2$, satisfies Condition 3. Consider a triple $x, y, z$ and suppose that there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $(x, y) \in R_1$, $(y, z) \in R_1^*$, and $(x, z) \notin R_1$. Then, $d_{\{1\}}(y, z) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) \neq \frac{1}{2} + d_{\{1\}}(x, z),$$

contradicting (8). Now, suppose that $(x, y) \in R_1^*$, $(y, z) \in R_1$, and $(x, z) \notin R_1$. Then, $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) \neq \frac{1}{2} + d_{\{1\}}(x, z),$$

contradicting (8). Hence, each pair $(R_i, R_i^*)$, $i = 1, 2$, satisfies Condition 4. We have proved that $\Omega$ is strictly decomposable. Conversely, suppose that $\Omega$ is strictly decomposable. Then, there exist four sets $R_1$, $R_2$, $R_1^*$, and $R_2^*$, with $R_i \subsetneqq NTR$, $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1, 2$, such that, for all $(x, y) \in NTR$, we have $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$; $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$; moreover, $R_i$, $i = 1, 2$, satisfies Condition 1; $R_i$ and $R_i^*$, $i = 1, 2$, satisfy Condition 2; each pair $(R_i, R_i^*)$, $i = 1, 2$, satisfies Conditions 3 and 4. Determine $d$ as follows. For each $(x, y) \in NTR$, let $d_\emptyset(x, y) = 0$, $d_E(x, y) = 1$; $d_{\{i\}}(x, y) = 1$ if and only if $(x, y) \in R_i$; $d_{\{i\}}(x, y) = \frac{1}{2}$ if and only if $(x, y) \in R_i^*$; $d_{\{i\}}(x, y) = 0$ if and only if, $(x, y) \notin R_i$ and $(x, y) \notin R_i^*$, for $i = 1, 2$. Then, $d$ satisfies (1) and (2) as, for all $(x, y) \in NTR$, $(x, y) \in R_1$ if and only if $(x, y) \notin R_1^*$ and $(y, x) \notin R_2$, $(x, y) \in R_1^*$ if and only if $(y, x) \in R_2^*$. Moreover, it can be shown that $d$ satisfies (5) and (7), by using the same arguments developed in the proof of Theorem 3. Consider a triple $x, y, z$ and suppose there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) \geq d_{\{1\}}(x, z).$$

Then, $(x, y) \in R_1^*$ and $(x, z) \notin R_1$, contradicting Condition 3. Therefore, $d$ satisfies (6). Consider a triple $x, y, z$ and suppose there exist $\mathbf{p}, \mathbf{q} \in \Omega$ satisfying $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$. Moreover, suppose that $d_{\{1\}}(x, y) = \frac{1}{2}$ and

$$d_{\{1\}}(x, y) + d_{\{1\}}(y, z) > \frac{1}{2} + d_{\{1\}}(x, z).$$

Consider the following two cases. First, $d_{\{1\}}(y,z) = 1$. Then, $(x,y) \in R_1^*$, $(y,z) \in R_1$, and $(x,z) \notin R_1$, contradicting Condition 4. Second, $d_{\{1\}}(y,z) = \frac{1}{2}$. Then, $(x,y) \in R_1^*$, $(y,z) \in R_1^*$, and $(x,z) \notin R_1^*$, contradicting Condition 2. Finally, suppose that $d_{\{1\}}(x,y) = \frac{1}{2}$ and

$$d_{\{1\}}(x,y) + d_{\{1\}}(y,z) < \frac{1}{2} + d_{\{1\}}(x,z).$$

Consider the following two cases. First, $d_{\{1\}}(y,z) = 0$. Then, $(z,y) \in R_2$, $(y,x) \in R_2^*$, and $(z,x) \notin R_2$, contradicting Condition 4. Second, $d_{\{1\}}(y,z) = \frac{1}{2}$. Then, $(x,y) \in R_1^*$, $(y,z) \in R_1^*$, and $(x,z) \notin R_1^*$, contradicting Condition 2. Therefore, $d$ satisfies (8). $d$ is nondictatorial as $\emptyset \subsetneqq R_i^* \subset NTR$, $i = 1,2$. Hence, $d$ is a nondictatorial ternary solution to IP1$'$ on $\Omega$. ■

Our characterization theorem for ASWFs with ties follows from Theorem 1 as a corollary of Theorem 5. This corollary is the generalization of Kalai and Muller's Theorem 2 for ASWFs without ties.

**Corollary 5.** *There exists a nondictatorial ASWF with ties on $\Omega$, $f$, for $n \geq 2$, if and only if $\Omega$ is strictly decomposable.*

**Proof**. It is an immediate consequence of Propositions 3 and 4, Theorems 1 and 5, and Corollary 4. ■

# 7 The relationship between decomposable and strictly decomposable domains

In this section, we analyze the relationship between the notions of decomposable and strictly decomposable domain. The following example illustrates the two notions.

**Example 1.** *Let $A = \{a,b,c,d\}$ and $\Omega = \{\mathbf{p} \in \Sigma : a\mathbf{p}b\mathbf{p}c\mathbf{p}d, c\mathbf{p}d\mathbf{p}a\mathbf{p}b, d\mathbf{p}c\mathbf{p}b\mathbf{p}a\}$. Then, $\Omega$ is decomposable and strictly decomposable.*

**Proof**. Let us notice that $NTR = \mathcal{A}^2$. The triples $x,y,z$ for which there exist $\mathbf{p},\mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$ are c,a,b; d,a,b; a,c,d; b,c,d. The triples $x,y,z$ for which there exist $\mathbf{p},\mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$ are a,b,c; a,b,d; a,c,d; b,c,d. Let $R_1 = \{(a,b),(b,a),(c,d),(d,c)\}$ and $R_2 = \{(a,c),(c,a),(a,d),(d,a),(b,c),(c,b),(b,d),(d,b)\}$. Then, we have $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$. Moreover, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$. $R_1$ vacuously satisfies Conditions 1 and 2. $R_2$ satisfies

Condition 1 as we have: $(a, c) \in R_2$ and $(a, d) \in R_2$; $(c, a) \in R_2$ and $(c, b) \in R_2$; $(d, a) \in R_2$ and $(d, b) \in R_2$; $(b, c) \in R_2$ and $(b, d) \in R_2$. $R_2$ vacuously satisfies Condition 2. We have shown that $\Omega$ is decomposable. Now, let $V_1 = \{(a, b), (c, d)\}$, $V_2 = \{(a, c), (c, a), (a, d), (d, a), (b, c), (c, b), (b, d), (d, b)\}$, $V_1^* = \{(b, a), (d, c)\}$, $V_2^* = \{(a, b), (c, d)\}$. Then, we have $V_i \subsetneqq NTR$, $i = 1, 2$, and $\emptyset \subsetneqq V_i^* \subset NTR$, $i = 1, 2$. Moreover, for all $(x, y) \in NTR$, we have: $(x, y) \in V_1$ if and only if $(x, y) \notin V_1^*$ and $(y, x) \notin V_2$; $(x, y) \in V_1^*$ if and only if $(y, x) \in V_2^*$. $V_1$ vacuously satisfies Conditions 1 and 2. $V_1^*$ vacuously satisfies Condition 2. Moreover, the pair $(V_1, V_1^*)$ vacuously satisfies Conditions 3 and 4. $V_2$ satisfies Conditions 1 and 2 as $V_2 = R_2$. $V_2^*$ vacuously satisfies Condition 2. The pair $(V_2, V_2^*)$ vacuously satisfies Condition 3. Moreover, it satisfies Condition 4 as we have: $(a, c) \in V_2$, $(c, d) \in V_2^*$, and $(a, d) \in V_2$; $(b, c) \in V_2$, $(c, d) \in V_2^*$, and $(b, d) \in V_2$; $(a, b) \in V_2^*$, $(b, c) \in V_2$, and $(a, c) \in V_2$; $(a, b) \in V_2^*$, $(b, d) \in V_2$, and $(a, d) \in V_2$. We have shown that $\Omega$ is strictly decomposable. ■

The example above specifies a domain which is both decomposable and strictly decomposable. Nonetheless, this is not the general case. In the following, we will show, with a theorem and a further example, that a strictly decomposable domain is always decomposable, but the converse is not true.

In order to obtain these results, we preliminarily show the following theorem on the nondictatorial solutions to IP1$'$.

**Theorem 6.** *If there exists a nondictatorial ternary solution to IP1$'$ on $\Omega$, $d$, for $n = 2$, then there exists a nondictatorial binary solution to IP1$'$ on $\Omega$, $\hat{d}$, for $n = 2$.*

**Proof**. Let $d$ be a ternary solution to IP1$'$ on $\Omega$, for $n = 2$. Determine $d'$ as follows. Consider $\mathbf{q} \in \Sigma$. For each $(x, y) \in NTR$, let: $d'_\emptyset(x, y) = 0$, $d'_E(x, y)$=1; $d'_{\{i\}}(x, y) = d_{\{i\}}(x, y)$, if $d_{\{i\}}(x, y) \in \{0, 1\}$, $i = 1, 2$; $d'_{\{1\}}(x, y) = 1$ and $d'_{\{2\}}(y, x) = 0$, if $d_{\{1\}}(x, y) = d_{\{2\}}(y, x) = \frac{1}{2}$ and $x\mathbf{q}y$. Then, it is immediate to verify that $d'$ is a solution to IP1$'$ on $\Omega$, for $n = 2$. Suppose that $d'$ is nondictatorial. Then, $\hat{d} = d'$ is a nondictatorial binary solution to IP1$'$ on $\Omega$, for $n = 2$. Suppose that $d'$ is dictatorial: say, for example, that, for all $(x, y) \in NTR$, $d_S(x, y) = 1$, for all $S$ containing agent 1. In this case, we can say that agent 1 is the dictator for $d'$. Determine $d''$ as follows. Let $\mathbf{q}^{-1} \in \Sigma$ be an antisymmetric preference ordering such that, for each $(x, y) \in \mathcal{A}^2$, $x\mathbf{q}y$ if and only if $y\mathbf{q}^{-1}x$. For each $(x, y) \in NTR$, let: $d''_\emptyset(x, y) = 0$, $d''_E(x, y)$=1; $d''_{\{i\}}(x, y) = d_{\{i\}}(x, y)$, if $d_{\{i\}}(x, y) \in \{0, 1\}$, $i = 1, 2$; $d''_{\{1\}}(x, y) = 1$ and $d''_{\{2\}}(y, x) = 0$, if $d_{\{1\}}(x, y) = d_{\{2\}}(y, x) = \frac{1}{2}$ and

$x\mathbf{q}^{-1}y$. Then, it is immediate to verify that $\hat{d} = d''$ is a binary solution to IP1$'$ on $\Omega$, for $n = 2$, and that agent 1 is not a dictator for $d''$. Suppose that agent 2 is a dictator for $d''$. Consider $(x, y) \in NTR$ such that $d_{\{1\}}(x, y) = d_{\{2\}}(y, x) = \frac{1}{2}$. Suppose that $y\mathbf{q}x$. This implies that $d'_{\{1\}}(x, y) = 0$ and agent 1 is not a dictator for $d'$, a contradiction. But then, we must have that $x\mathbf{q}y$. Consider variables $d_{\{1\}}(y, x)$ and $d_{\{2\}}(x, y)$. Suppose that $d_{\{1\}}(y, x) = 1$ and $d_{\{2\}}(x, y) = 0$. Then, agent 2 is not a dictator for $d''$, a contradiction. Suppose that $d_{\{1\}}(y, x) = 0$ and $d_{\{2\}}(x, y) = 1$. Then, agent 1 is not a dictator for $d'$. This implies that $d_{\{1\}}(y, x) = d_{\{2\}}(x, y) = \frac{1}{2}$ and this, in turn, implies that $d''_{\{2\}}(x, y) = 0$ and agent 2 is not a dictator of $d''$, a contradiction. Then, $\hat{d} = d''$ is a nondictatorial binary solution to IP1$'$ on $\Omega$, for $n = 2$. ■

Again, we straightforwardly obtain a correspondent result for nondictatorial ASWFs as a corollary of Theorem 6. A first proof of this result is due to Maskin (1979).

**Corollary 6.** *If there exists a nondictatorial ASWF with ties on $\Omega$, $f$, for $n \geq 2$, then there exists a nondictatorial ASWF without ties on $\Omega$, $\hat{f}$, for $n \geq 2$.*

**Proof**. It is an immediate consequence of Propositions 3 and 4, and of Theorems 1, 2, 4, and 6.[4] ■

On the basis of the previous results, the following theorem can be immediately proved.

**Theorem 7.** *If a domain $\Omega$ is strictly decomposable, then it is decomposable.*

**Proof**. Let $\Omega$ be a strictly decomposable domain. Then, by Theorem 5, there exists a nondictatorial ternary solution to IP1$'$ on $\Omega$, $d$, for $n = 2$. But then, by Theorem 6, there exists a nondictatorial binary solution to IP1$'$ on $\Omega$, $\hat{d}$, for $n = 2$. Hence, by Theorem 3, $\Omega$ is decomposable. ■

The following example shows that the converse of Theorem 7 does not hold.

**Example 2.** *Let $A = \{a, b, c, d\}$ and $\Omega = \{\mathbf{p} \in \Sigma :\ a\mathbf{p}b\mathbf{p}c\mathbf{p}d, c\mathbf{p}a\mathbf{p}d\mathbf{p}b, d\mathbf{p}c\mathbf{p}b\mathbf{p}a, b\mathbf{p}d\mathbf{p}a\mathbf{p}c\}$. Then, $\Omega$ is decomposable but it is not strictly decomposable.*

[4] We notice that our version of Maskin's Theorem 3 does not cover the case where $\Omega \cap (\mathcal{R} \setminus \mathcal{P}) \neq \emptyset$.

**Proof**. Let us notice that $NTR = \mathcal{A}^2$. The triples $x, y, z$ for which there exist $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$ are: c,a,b; c,b,a; a,b,d; a,d,b,; d,a,c; d,c,a; b,c,d; b,d,c. The triples $x, y, z$ for which there exist $\mathbf{p}, \mathbf{q} \in \Omega$ such that $x\mathbf{p}y\mathbf{p}z$ and $z\mathbf{q}y\mathbf{q}x$ are: a,b,c; c,a,b; a,b,d; a,d,b; a,c,d; c,a,d; b,c,d; c,d,b. Let $R_i = \{(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)\}$, $i = 1,2$. Then, we have $\emptyset \subsetneqq R_i \subsetneqq NTR$, $i = 1,2$. Moreover, for all $(x,y) \in NTR$, we have $(x,y) \in R_1$ if and only if $(y,x) \notin R_2$. $R_i$ satisfies Condition 1, $i = 1,2$, as we have: $(a,b) \in R_i$ and $(a,d) \in R_i$; $(a,d) \in R_i$ and $(a,b) \in R_i$; $(b,c) \in R_i$ and $(b,d) \in R_i$; $(b,d) \in R_i$ and $(b,c) \in R_i$, $i = 1,2$. $R_i$ satisfies Condition 2, $i = 1,2$, as we have: $(a,b) \in R_i$, $(b,c) \in R_i$, and $(a,c) \in R_i$; $(a,b) \in R_i$, $(b,d) \in R_i$, and $(a,d) \in R_i$; $(a,c) \in R_i$, $(c,d) \in R_i$, and $(a,d) \in R_i$; $(b,c) \in R_i$, $(c,d) \in R_i$, and $(b,d) \in R_i$, $i = 1,2$. We have shown that $\Omega$ is decomposable. Now suppose that $\Omega$ is strictly decomposable. Then, there exist four sets $V_1$, $V_2$, $V_1^*$, and $V_2^*$, with $V_i \subsetneqq NTR$, $\emptyset \subsetneqq V_i^* \subset NTR$, $i = 1,2$, such that, for all $(x,y) \in NTR$, we have: $(x,y) \in V_1$ if and only if $(x,y) \notin V_1^*$ and $(y,x) \notin V_2$; $(x,y) \in V_1^*$ if and only if $(y,x) \in V_2^*$. Moreover, $V_i$, $i = 1,2$, satisfies Condition 1; $V_i$ and $V_i^*$, $i = 1,2$, satisfy Condition 2; each pair ($V_i$, $V_i^*$), $i = 1,2$, satisfies Conditions 3 and 4. Suppose that $(a,b) \in V_1^*$ and $(b,a) \in V_2^*$. Then, $(a,d) \in V_1$ as the pair $(V_1, V_1^*)$ satisfies Condition 3. But then, $(a,b) \in V_1$ as $V_1$ satisfies Condition 1, a contradiction. Suppose that $(a,c) \in V_1^*$ and $(c,a) \in V_2^*$. Then, $(c,b) \in V_2$ as the pair $(V_2, V_2^*)$ satisfies Condition 3. But then, $(c,a) \in V_2$ as $V_2$ satisfies Condition 1, a contradiction. Suppose that $(a,d) \in V_1^*$ and $(d,a) \in V_2^*$. Then, $(a,b) \in V_1$ as the pair $(V_1, V_1^*)$ satisfies Condition 3. But then, $(a,d) \in V_1$ as $V_1$ satisfies Condition 1, a contradiction. Suppose that $(b,c) \in V_1^*$ and $(c,b) \in V_2^*$. Then, $(b,d) \in V_1$ as the pair $(V_1, V_1^*)$ satisfies Condition 3. But then, $(b,c) \in V_1$ as $V_1$ satisfies Condition 1, a contradiction. Suppose that $(b,d) \in V_1^*$ and $(d,b) \in V_2^*$. Then, $(b,c) \in V_1$ as the pair $(V_1, V_1^*)$ satisfies Condition 3. But then, $(b,d) \in V_1$ as $V_1$ satisfies Condition 1, a contradiction. Suppose that $(c,d) \in V_1^*$ and $(d,c) \in V_2^*$. Then, $(d,a) \in V_2$ as the pair $(V_2, V_2^*)$ satisfies Condition 3. But then, $(d,c) \in V_2$ as $V_2$ satisfies Condition 1, a contradiction. Hence, $V_i^* = \emptyset$, $i = 1,2$, a contradiction. We have shown that $\Omega$ is not strictly decomposable. ■

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