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An Early Existence Proof in Consumer Theory: A Critical Reassessment of Luigi Amoroso (1928)

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Abstract

This paper critically re-examines Amoroso's (1928) attempt to prove the existence of consumer equilibrium. We identify lacunae and analytical flaws in his proof and establish limits to the validity of his asserted theorem. The underlying difficulty is that the coordinatewise boundary conditions he imposes on marginal utilities need not control simultaneous approaches to the boundary along the budget constraint. Nevertheless, we show that his proof strategy, based on the intermediate value theorem and on induction on the number of commodities, can be completed under the additional assumption that the utility function is additively separable. We also show that Amoroso's equilibrium equations arise as a special case of the system that Wald (1936) studied in his analysis of competitive equilibrium in a production economy. The historical literature has credited Amoroso with anticipating Wald in adopting an explicitly axiomatic formulation. We argue that the anticipation extends to the proof strategy: Amoroso's argument already contains two principal components of Wald's later proof, namely the intermediate value argument and induction on the number of commodities. Yet this anticipation of method must be distinguished from a successful proof of the general result announced in 1928. The mathematical analysis thus refines the historical attribution by distinguishing a valid restricted result, an unsuccessful general claim, and a proof strategy that becomes viable under additional assumptions.

Keywords: Luigi Amoroso; consumer equilibrium; existence proofs; additive separability; intermediate value theorem; Abraham Wald.

JEL Codes: B23, C62, D11

MSC: 91-03

1 Introduction

Guccione and Minelli (1999, p. 589), aiming "to make historians of thought aware of an early example of the proper use of mathematics in economics," discuss

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the significance of Luigi Amoroso's 1928 paper entitled *Discussione del sistema di equazioni che definiscono l'equilibrio del consumatore*,[1] since, with the notable exception of Joseph A. Schumpeter, this contribution was largely ignored by scholars. In his article Amoroso sought to prove the existence of a consumer equilibrium, that is, the existence of a commodity bundle that maximizes the consumer's utility function within a (non-closed) budget set.[2] Guccione and Minelli frame the methodological and historical importance of Amoroso (1928) mainly in relation to Schumpeter's (1954) favorable assessment of the work and to the later claim of Hildenbrand (1983) that Wald (1935, 1936) represented the first application of the axiomatic approach in economic theory.[3] According to them, Amoroso's paper is important not only because it anticipated much later work in demand theory, but also because it is one of the earliest applications of the axiomatic approach in economics, in the sense of Debreu (1984).

Earlier, Guerraggio (1990) and Legris (1995) had already recognized Amoroso's contribution as an anticipation of Wald as regards the axiomatic approach in economics. Guerraggio outlines the four lemmas that form the backbone of Amoroso's article, treats the existence argument in Lemmas III and IV as successful, and regards the inaccuracies identified in Lemmas I and II as marginal—rightly so, one is inclined to say. He then concludes:

> È più importante osservare come l'impostazione di questa Nota, in ciò accomunata ai corrispondenti articoli di Wald, sia esplicitamente assiomatica: le ipotesi vengono elencate con chiarezza all'inizio del procedimento e la loro motivazione economica, per lo meno, non è più rilevante di quella analitica. (Guerraggio 1990, p. 32)[4]

Legris likewise emphasizes the anticipatory character of the contribution and its failure to generate further research in Italy. Both assessments leave room for a closer examination of what Amoroso's argument actually establishes, and of the precise sense in which it anticipates Wald.

More recently, Pomini (2022) linked Amoroso's contribution to the debate on what mathematicians at the time—Gaetano Scorza and Ugo Broggi being the main examples—regarded as the insufficient mathematical rigor of Walras's and Pareto's theories. To quote Pomini (2022, p. 52): "Unlike Pareto, Amoroso did not ignore the mathematical objections of the mathematicians and provided a single partial answer in his 1928 article."[5] Indeed, Amoroso explicitly acknowledges that the

[1] *An Analysis of the System of Equations Defining Consumer Behaviour* as translated in Pasinetti (1992). All English translations of Amoroso (1928) are taken from that volume.

[2] The non-closedness of the budget set is not analytically innocuous, since it precludes a plain application of Weierstrass' Theorem. It stems from Amoroso's choice of an open consumption set, which at the same time provides a natural setting for differentiable demand theory.

[3] The same view is expressed by Mutoh (2003, p. 167), who writes: "The publication of these two papers [Wald (1935, 1936)] should be regarded as a starting point of the axiomatization in economics."

[4] "It is more important to observe that the approach of this Note, in this respect akin to the corresponding articles by Wald, is explicitly axiomatic: the hypotheses are listed clearly at the outset of the argument, and their economic motivation, at least, is no more relevant than their analytical one." (Our translation).

[5] Among the criticisms raised by mathematicians, the one most relevant to the subject of this article is that merely counting equations and unknowns is not sufficient to establish the existence

classical theory of consumer behavior, as formulated by Walras, Edgeworth, and Pareto, remains incomplete, and sets out to fill this gap in his article.

> La teoria classica dell'equilibrio del consumatore, nella forma ben nota quale è stata costruita dal Walras, dall'Edgeworth, dal Pareto, e quale è riportata nei trattati, comprese le mie lezioni di Economia matematica, non è completa. Essa invero, riducendo, il problema ad un problema di massimo, impone ai dati di soddisfare a certe equazioni, le quali sono necessarie per il massimo, non già necessarie e sufficienti. Manca poi la discussione del sistema, cioè la dimostrazione della esistenza di soluzioni, la determinazione del loro numero, ecc.
>
> Colmare questa lacuna è oggetto della presente Nota. (Amoroso 1928, p. 31)[6]

Our contribution is to distinguish the anticipation of an axiomatic approach from the anticipation of a specific proof strategy, and both from the successful establishment of an existence result, the latter being what was needed to answer the objection raised by the Italian mathematicians mentioned above. We do so by identifying in Amoroso's argument two principal components subsequently employed by Wald—an intermediate value argument[7] and induction on the number of commodities—and by determining the limits of their validity under Amoroso's stated assumptions. This mathematical examination makes the historical assessment more precise: it establishes which parts of the argument can be completed, where the asserted result is false, and under which additional assumptions the proposed strategy becomes viable.

To this end, the reconstruction of Wald's proof by Hildenbrand (1998) is helpful, as it shows that Wald's approach was essentially based on the intermediate value theorem and induction on the number of commodities. In an attempt to make Amoroso's role as a forerunner fully transparent, we initially set out to reconstruct his proof in the spirit of Hildenbrand's (1998) exposition of Wald's argument. However, we found that the proof of Amoroso's Lemma III, in which the intermediate value theorem is applied to establish the existence of a consumer equilibrium in the two-commodity case (the base for the induction), contains analytical gaps and flaws that undermine its validity, as well as that of the subsequent Lemma IV and, ultimately, of the existence part of the *Teorema fondamentale* (Fundamental Theorem), which Amoroso (1928) aims to prove. This is why our reconstruction focuses first on Amoroso's Lemma III.

We show that the result of this lemma can be recovered once an additional assumption is introduced requiring that the utility function be additively separable.

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and uniqueness of a solution to a system of equations; see Fernández López (2003).

[6] "The classical theory of consumer behaviour, in the well-known formulation by Walras, Edgeworth and Pareto, and given in my lessons for Mathematical Economics, is not complete. Indeed, in reducing the problem to one of a maximum, it restricts the given quantities to satisfy certain equations which are necessary, but not necessary and sufficient for a maximum. An analysis of the system is required, showing that solutions do exist, the number of solutions, etc. The present note intends to provide such an analysis."

[7] The intermediate value theorem is also known as Bolzano's theorem, after Bernard Bolzano, who proved it in his 1817 monograph.

Building on Amoroso's own proof strategy, we then prove that, under this assumption, a consumer equilibrium exists for an arbitrary number of commodities—a result amending Amoroso's original argument. We emphasize that additive separability is never mentioned in Amoroso (1928), which, on the contrary, lays claim to generality. Nor should additive separability be understood as a necessary condition. Amoroso's Fundamental Theorem fails in the generality he claimed because the coordinatewise boundary conditions he assumes need not control the simultaneous approaches to the boundary along the budget constraint, whereas additive separability removes this difficulty.

We also find that Amoroso's system of equations is a special case of the one considered by Wald (1936). Furthermore, under the restriction of additive separability, Amoroso's assumptions imply Wald's assumptions, which are sufficient for existence. We thus draw a direct link between the two contributions and make it evident that, mathematically, Wald's system of equations is a generalization of Amoroso's. Without additive separability, however, Amoroso's coordinatewise boundary conditions do not imply the boundary condition on which Wald's inductive step relies, and it is this condition, rather than the proof strategy, that separates the two arguments.

Our results fit within two major lines of inquiry. The first concerns the genesis of existence proofs for competitive equilibrium, since the analysis of consumer equilibrium can be regarded as preliminary to that of competitive equilibrium. The second investigates the early use of classical mathematical results in economics. On the first, the seminal contribution is Weintraub (1983), recently complemented by Düppe and Weintraub (2014), on the origin of the 1954 existence proofs by Kenneth Arrow, Gérard Debreu, and Lionel McKenzie, and by Düppe and Weintraub (2016), on the evidence concerning a lost fixed-point proof for an exchange economy that Wald presented in 1935.[8] On the second, Creedy (1980) examines early applications of Lagrange multipliers and the mathematical difficulties accompanying them. He notes, in particular, that Edgeworth did not extend his discussion of implicit equations to the solution of a simultaneous system in several variables (p. 374n2). Interestingly, Amoroso confronts a difficulty of the same order: the multiplier method delivers conditions that are necessary for a maximum, but no assurance that any bundle satisfies them. Supplying that assurance is the task he sets himself. Although his proof does not establish the result in the claimed generality, this aim is central to its historical significance. More recently, Ghosh et al. (2022) explore the applications and the many far-reaching consequences of the intermediate value theorem across economic theory. By an application of the theorem we mean here an explicit and analytically grounded use, rather than an intuitive assertion, such as a Cournot-style argument that two continuous curves must intersect under appropriate conditions. From this perspective, Amoroso's attempted existence proof of 1928 constitutes an early—possibly the first—application of the intermediate value theorem to an existence problem in economics.

These two lines of inquiry are less distant than they may appear. Takayama (1985, p. 276n19) observes: "[McKenzie (1954)] is probably the first article in economics to use Kakutani's fixed point theorem." In retrospect, the relationship between fixed-point methods and the intermediate value arguments considered

[8] That proof is distinct from the published production-economy proofs considered here; our comparison concerns the latter's combination of induction and intermediate value arguments.

here can be made precise: Miranda (1940) established the equivalence of Brouwer's fixed-point theorem and a multidimensional version of Bolzano's theorem, while Hildenbrand (1998, p. 59) points out: "Wald's intermediate value theorem for a correspondence [...] is, of course, a one dimensional version of Kakutani's fixed point theorem." Tracing the early use of intermediate value arguments is therefore not a marginal appendix to the history of existence results based on fixed-point methods, but part of the same history.

Our critical re-examination of Amoroso (1928) is organized as follows. In Section 2, we state the consumer maximization problem studied by Amoroso and his assumptions. It is important to remark that Amoroso assumed a bounded and open consumption set, and this is what makes the budget set non-closed and the problem interesting. In Section 3, we focus on his Lemma III, show that there are instances in which no consumer equilibrium exists in the two-commodity case, and supply some missing arguments. In Section 4, we extend Amoroso's analysis in two directions. First, we allow for utility functions of arbitrary sign, dropping Amoroso's restriction to functions that are everywhere positive on their domain; as we document, this restriction is a stable feature of his work, inherited from Pareto. Second, we replace the bounded consumption set with the now standard strictly positive orthant. In this extended setting, the intermediate value theorem can be applied in the two-commodity case, so that a consumer equilibrium exists and the base for the induction is available. However, we also provide a counterexample to existence in the three-commodity case, which implies that the inductive step fails. Our analysis thus shows that, although Amoroso set out the same proof strategy later followed by Wald, this strategy does not work within his own framework. For this reason, in Section 5, we introduce the assumption of an additively separable utility function. Under this additional restriction, we first establish the existence of a consumer equilibrium for an arbitrary number of commodities, both in Amoroso's original bounded setting and in our unbounded extension, by means of a direct argument that does not require induction. We then argue that Amoroso's own proof strategy, based on the intermediate value theorem and induction on the number of commodities, can be completed once the cases he overlooked are taken into account. In Section 6, we show that Amoroso's equilibrium equations can be obtained as a special case of the system of equations considered by Wald (1936), and we compare the two sets of assumptions. In Section 7, we draw some conclusions. All the proofs are collected in the Appendix.

2 Amoroso's consumer equilibrium existence problem

The purpose of Amoroso's work was to fill a significant gap in the classical theory of consumer equilibrium: the lack of both a formal proof of existence and sufficient conditions for such a result.

With this in mind, Amoroso considers the consumer problem of choosing a commodity bundle $\mathbf{x} = (x_1, \ldots, x_n)$ that maximizes $\Phi : D \to \mathbb{R}$, what we nowadays call a utility function,[9] on the budget hyperplane determined by the initial endowment

[9] In Amoroso (1921, p. 66) and Amoroso (1942, p. 11), the function Φ is called *ofelimità totale* or simply *ofelimità*, as the two passages quoted in Section 4 show. Broadly speaking, *ofelimità* refers

$\mathbf{c} = (c_1, \ldots, c_n)$ and the price vector $\mathbf{p} = (p_1, \ldots, p_n)$. Formally, he deals with the following maximization problem

$$\begin{aligned} \max_{\mathbf{x} \in D} \quad & \Phi(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{p} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{c} \end{aligned} \tag{1}$$

and aims to prove the existence and uniqueness of its solution, $\mathbf{x}^*$, given Φ, $\mathbf{c}$, and $\mathbf{p}$. The assumptions made by Amoroso are denoted hereafter by (A_0)–(A_7).

(A_0) The consumption set is an open subset D of $\mathbb{R}^n$, defined as the following Cartesian product

$$D = \prod_{i=1}^{n}]a_i, b_i[,$$

where n is the number of commodities, and a_i, b_i are the components of two vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, satisfying $a_i < b_i$ and $0 < b_i < \infty$, for each $i = 1, \ldots, n$.

(A_1) The price vector $\mathbf{p}$ and the consumer's initial endowment $\mathbf{c}$ are such that $\mathbf{p} \in \mathbb{R}^n_{++}$ and $\mathbf{c} \in D$.

Amoroso originally focused on a bounded consumption set, thereby excluding the now standard case where the consumption set corresponds to the strictly positive orthant $\mathbb{R}^n_{++}$. In Section 4, we consider the latter by extending the analysis of Amoroso to the case in which $a_i = 0$ and $b_i = \infty$ for each $i = 1, \ldots, n$. We also note that the maximization problem is defined over an open set, which is why it is natural to use the Lagrange method to determine $\mathbf{x}^*$, if it exists.

(A_2) The utility function is positive on D, *i.e.*, $\Phi(\mathbf{x}) > 0$ for each $\mathbf{x} \in D$.

(A_3) The utility function is twice continuously differentiable on D, *i.e.*, $\Phi \in C^2(D)$.

(A_4) The partial derivatives of Φ are positive on D, *i.e.*, $\Phi_{x_i}(\mathbf{x}) > 0$ for each $\mathbf{x} \in D$ and for each $i = 1, \ldots, n$.[10]

(A_5) The Hessian matrix of Φ is negative definite on D.

Amoroso's choice of an open consumption set is well suited to assumption (A_3). Indeed, the standard definition of differentiability is formulated on open sets, and this setting provides a natural framework for working with marginal utilities and studying differentiable demand functions, an approach that, as Barten and Böhm (1982) remark, was the classical one.[11] Note also that assumptions (A_4) and (A_5)

to cardinal utility, whereas *indice dell'ofelimità* refers to ordinal utility (see, for instance, Pareto 1906, ch. III, §54n). In Amoroso (1928, p. 31), however, Φ is called *indice dell'ofelimità*, even though its positivity is assumed, a restriction that carries content only under a cardinal reading. We therefore read the term as a terminological slip rather than as an ordinal interpretation of Φ. For simplicity, we refer to Φ as the utility function throughout the paper.

[10] We denote by $\Phi_{x_i}(\mathbf{x})$ the partial derivative $\frac{\partial \Phi(\mathbf{x})}{\partial x_i}$.

[11] See Barten and Böhm (1982, secs. 11–12) and Guccione and Minelli (1999, p. 588n3). When Amoroso was writing, the theory of differentiability on non-open domains had significant antecedents but was still developing. Substantial advances followed in the 1930s, particularly in the study of the geometric conditions for the existence and uniqueness of the differential; see Dolecki and Greco (2011).

imply that the utility function is strictly increasing and strictly concave. Furthermore, since the constraint is linear, assumption (A_5) ensures that the Lagrange conditions are sufficient for optimality, as Amoroso himself shows in his Lemma II. Therefore, the maximization problem in (1) is equivalent to the following system of equations

$$\begin{cases} p_2\Phi_{x_1}(\mathbf{x}) - p_1\Phi_{x_2}(\mathbf{x}) = 0 \\ \dots \\ p_n\Phi_{x_1}(\mathbf{x}) - p_1\Phi_{x_n}(\mathbf{x}) = 0 \\ \mathbf{p}\cdot\mathbf{x} = H, \end{cases} \tag{2}$$

where $H = \mathbf{p}\cdot\mathbf{c}$ is the consumer's wealth and $H \in]\mathbf{p}\cdot\mathbf{a}, \mathbf{p}\cdot\mathbf{b}[$ by assumption (A_1). To introduce Amoroso's last assumptions, we define $D_{-j} = \prod_{i\neq j}]a_i, b_i[$ and use the somewhat standard notation for vectors $\mathbf{x} = (x_j, x_{-j})$, where $x_{-j} = (x_i)_{i\neq j}$.

(A_6) $\lim_{s\to a_i} \Phi_{x_i}(s, x_{-i}) = \infty$ for each $x_{-i} \in D_{-i}$ and for each $i = 1, \dots, n$.

(A_7) $\lim_{s\to b_i} \Phi_{x_i}(s, x_{-i}) = 0$ for each $x_{-i} \in D_{-i}$ and for each $i = 1, \dots, n$.

These last two assumptions can immediately be recognized as *ante litteram* Inada-style conditions. In the rest of the paper, we write $\Phi \in \mathcal{F}_+(n)$ whenever a function Φ satisfies assumptions (A_0) and (A_2)–(A_7). We write $\Phi \in \mathcal{F}(n)$ when all these assumptions except (A_2) are satisfied, that is, when the positivity of Φ on the entire domain is not assumed.

Amoroso focuses on the system of equations in (2) and aims to prove existence and uniqueness of a solution for every price vector $\mathbf{p} \in \mathbb{R}^n_{++}$ and every wealth $H \in]\mathbf{p}\cdot\mathbf{a}, \mathbf{p}\cdot\mathbf{b}[$. He first addresses uniqueness in his Lemma I. However, since uniqueness follows immediately from the strict concavity of Φ, the main challenge is to prove existence. Amoroso starts his analysis by proving the existence of a solution to the system of equations (2) in the two-commodity case, *i.e.*, $n = 2$. For higher dimensions, he uses induction on n to establish his main result, which he calls the "Fundamental Theorem."

Theorem (Amoroso's Fundamental Theorem)**.** *For any number n of commodities, if $\Phi \in \mathcal{F}_+(n)$, $\mathbf{p} \in \mathbb{R}^n_{++}$, and $H \in]\mathbf{p}\cdot\mathbf{a}, \mathbf{p}\cdot\mathbf{b}[$, the system of equations (2) has a unique solution.*

Amoroso's Fundamental Theorem (henceforth AFT) seeks to establish the existence result, thereby lending consistency to the notion of consumer equilibrium. Unfortunately, Amoroso's analysis has some inaccuracies and there are instances satisfying assumptions (A_0)–(A_7) in which the AFT is false.

3 A reconstruction of Amoroso's Lemma III

In this section we provide a detailed reconstruction of Amoroso's Lemma III, revealing some *lacunae* and analytical flaws that lead us to identify cases where the AFT does not hold by providing counterexamples to it. Throughout this section the consumption set is bounded, as in Amoroso's original setting.

Lemma III asserts the existence of a solution to the system of equations (2) in the two-commodity case, which is also the base for the induction. Hence, the problems we find in Lemma III not only invalidate it but also make it impossible to apply induction on the number of commodities, based solely on assumptions (A_0)–(A_7), as Amoroso erroneously proposes in his Lemma IV. This explains why we focus on Lemma III.

To simplify the analysis, without loss of generality, we set commodity 1 as the numeraire, denote the price of commodity 2 by P, and assume $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$, *i.e.,* $D =]0, 1[^2$. Amoroso's problem, as outlined in (2), then becomes the following: given a utility function $\Phi \in \mathcal{F}_+(2)$ defined over D and given positive real numbers H and P with $0 < H < 1 + P$, show the existence of $(x_1, x_2) \in D$ such that

$$\begin{cases} x_1 + Px_2 = H \\ P\Phi_{x_1}(x_1, x_2) - \Phi_{x_2}(x_1, x_2) = 0. \end{cases} \tag{3}$$

At this point the first problem in Amoroso's paper arises. On the last line of p. 37 (last line in Pasinetti 1992, p. 104) he writes the inequalities $a_2 < \xi_2 < b_2$ and $a_1 < \eta_1 < b_1$,[12] and claims that they always hold. This claim, however, is not correct: under our simplifications the two inequalities become

$$0 < \frac{H}{P} < 1 \quad \text{and} \quad 0 < H < 1, \tag{4}$$

and they fail in general, since the positive numbers H and P are only constrained by $0 < H < 1 + P$. A close inspection reveals that the analysis depends on the position of the budget line $x_1 + Px_2 = H$ relative to the bounded consumption set D. More specifically, four cases arise, according to whether $H < 1$ or $H \geq 1$ and whether $H < P$ or $H \geq P$. This mistake leads Amoroso to restrict his analysis to the case in Figure 1, where inequalities (4) are indeed true, thereby overlooking the three alternative cases shown in Figures 2–4. Notably, it is in such ignored cases that the AFT fails, as shown by Proposition 3 below.

Moreover, the inductive step of Lemma IV is itself flawed. Amoroso (1928, p. 39) asserts "Anche h_{n-1} verificherà allora alle (16), posto ivi $m = n - 1$,"[13] which fails in general, for the same reason as in the two-commodity case: the n-commodity analogue of the configurations in Figures 2–4 may occur.

Our reconstruction of Amoroso's proof starts from the only case considered by him. By observing Figure 1, we clearly see that he is facing the problem of finding a commodity bundle (x_1, x_2) on the segment $\overline{MN}$, with the endpoints $M = (0, H/P)$ and $N = (H, 0)$ excluded, where

$$P\,\Phi_{x_1}(x_1, x_2) - \Phi_{x_2}(x_1, x_2) = 0.$$

To simplify the subsequent analysis, let us introduce the following additional notation

$$x_s = H - Ps \text{ for } s \in]0, H/P[\quad \text{and} \quad g(s) = P\Phi_{x_1}(x_s, s) - \Phi_{x_2}(x_s, s).$$

[12] In Amoroso's notation, (ξ_1, ξ_2) and (η_1, η_2) are, respectively, the intersections of the budget line with the vertical line $\xi_1 = a_1$ and with the horizontal line $\eta_2 = a_2$.

[13] "Also h_{n-1} will satisfy (16), where $m = n - 1$."

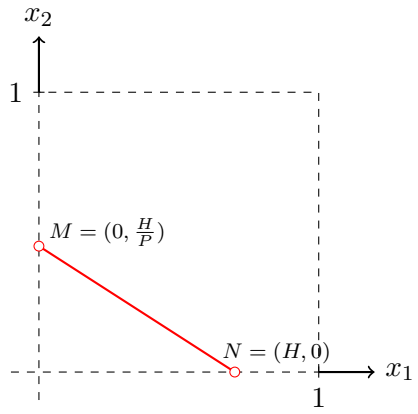


Figure 1: $H < 1$ and $H < P$

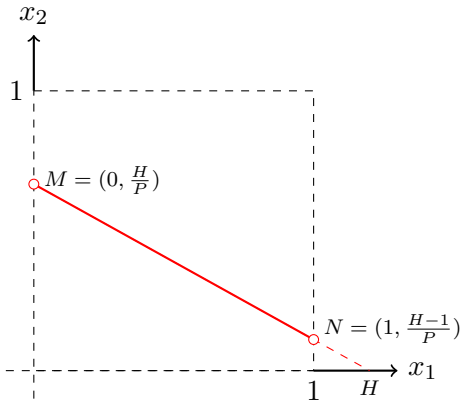


Figure 2: $H \geq 1$ and $H < P$

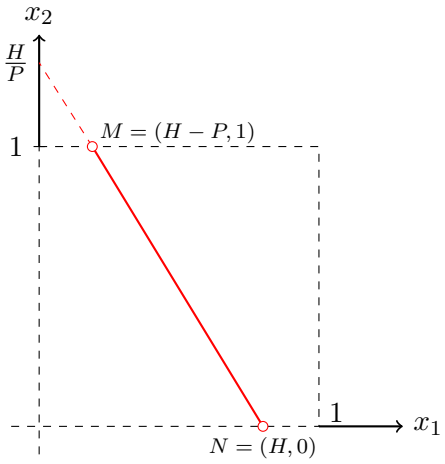


Figure 3: $H < 1$ and $H \geq P$

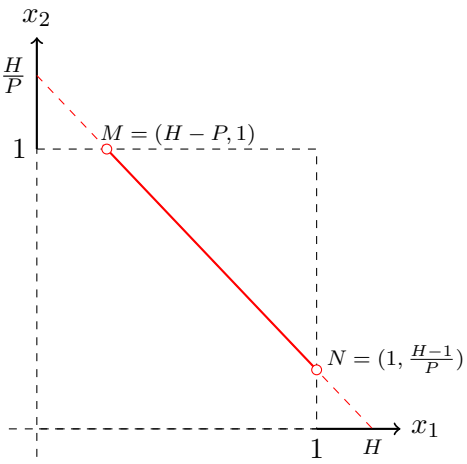


Figure 4: $H \geq 1$ and $H \geq P$

Amoroso's problem is thus equivalent to finding a point $s \in]0, H/P[$ such that $g(s) = 0$. Amoroso proposes a hasty argument to claim that $\lim_{s\to 0} \Phi_{x_2}(x_s, s) = \infty$, appealing only to assumption (A_6), and that $\Phi_{x_1}(x_s, s)$ is bounded. Thus, he can readily conclude that

$$\lim_{s\to 0} g(s) = \lim_{s\to 0} P\Phi_{x_1}(x_s, s) - \lim_{s\to 0} \Phi_{x_2}(x_s, s) = -\infty. \tag{5}$$

By following *mutatis mutandis* the same steps, he concludes that $\lim_{s\to \frac{H}{P}} g(s) = \infty$. Then, he derives the existence of a root of $g(s) = 0$ by the intermediate value theorem.

Unfortunately, the argument just outlined contains two significant omissions. The first is that the pointwise condition (A_6), $\lim_{s\to 0} \Phi_{x_2}(x_1, s) = \infty$ for each $x_1 \in]0, 1[$, does not by itself imply that $\lim_{s\to 0} \Phi_{x_2}(x_s, s) = \infty$. Nonetheless, it can be shown that, under the full set of assumptions made above, this limit does hold whenever $H < 1$. The next proposition formalizes this point, showing, more precisely, that assumptions (A_5) and (A_6) alone suffice.

Proposition 1. *Assume that $H < 1$ and that $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_5) and (A_6). Then,*

$$\lim_{s\to 0} \Phi_{x_2}(x_s, s) = \infty.$$

Thus, although Amoroso's argument is superficial, his claim is true for $H < 1$ provided that one carefully uses the fact that the utility function satisfies assumption (A_5). Strict concavity therefore plays a role not only in uniqueness but also, crucially, in existence. The proof of Proposition 1, which can be found in Appendix A.1, supplies the arguments missing from Amoroso's analysis. We also remark that the proof relies on Dini's theorem, which states that a monotone sequence of continuous functions on a compact set that converges pointwise to a continuous function converges uniformly. The second omission concerns the claim "l'altra derivata $\frac{\partial \Phi}{\partial x_1}$ resta finita" (Amoroso, 1928, p. 38),[14] meaning that Φ_{x_1} is bounded. The following example (details in Appendix A.2) shows that the statement—however natural it may seem—is at least open to question.

Example 1. *The following utility function*

$$\Phi(x_1, x_2) = 2\sqrt{x_1} - x_1 + 2\sqrt{x_2} - x_2 - (1 - x_1)^2 \left(x_2^{-1} + x_2\right),$$

defined on the consumption set $]0,1[^2$, belongs to $\mathcal{F}(2)$. Moreover for every $H \in]0,1[$ and for every P

$$\lim_{s\to 0} \Phi_{x_1}(x_s, s) = \infty.$$

In this example, Φ is not everywhere positive, so Amoroso's claim is not refuted. However, positivity is the only one of Amoroso's assumptions that is not satisfied, and Amoroso gives no argument for why (A_2) should make Φ_{x_1} bounded above. This issue is addressed in the following proposition, whose proof in Appendix A.3 once again highlights the role of the negative definiteness of the Hessian of Φ, which otherwise appears to serve only to ensure uniqueness.

Proposition 2. *Assume that $H < 1$ and that $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_2) and (A_5). Then, $\Phi_{x_1}(x_s, s)$ is bounded above on every interval $]0, s_0[$ with $s_0 < \min\{\frac{H}{P}, 1\}$.*

In light of Propositions 1 and 2, Amoroso's argument, as presented above, acquires a rigorous foundation and yields the following theorem.

Theorem 1 (Two-commodity AFT)**.** *For two commodities, if $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_2), (A_5), and (A_6), $0 < H < 1$, and $H < P$, then the system of equations (3) has a unique solution.*

It is worth mentioning that assumption (A_7) plays no role in the validity of the above theorem, as the proofs of the previous propositions make clear. Theorem 1 establishes Lemma III in the case of Figure 1, which is also the only case for which Amoroso presents a proof. Consequently, a further flaw arises: the way he states Lemma III leaves no doubt that he intends it to hold in the remaining cases, Figures 2–4, as well. However, as the following proposition shows, it does not.

[14] "While the other derivative will remain finite."

Proposition 3 (Failure of AFT)**.** *For two commodities, if either $H \geq 1$ or $H \geq P$, there exists at least one function $\Phi \in \mathcal{F}_+(2)$ for which the system of equations (3) has no solution.*

The proof consists in constructing counterexamples for the cases of Figures 2–4. For instance, for $H = P = 1$, which is the case in Figure 4, the function

$$\Phi(x_1, x_2) = \frac{1}{21}(1 - x_2)^2(2\sqrt{x_1} - x_1) + 2\sqrt{x_2} - x_2$$

is such that $\Phi \in \mathcal{F}_+(2)$ and $g(s) < 0$ for each $s \in]0, 1[$. Therefore, no solution exists to the system of equations (3). The detailed calculations can be found in Appendix A.4, where all cases are considered in a more general setting. This example illustrates a limitation of the coordinatewise boundary conditions. Assumption (A_6) requires $\Phi_{x_1}(x_1, x_2)$ to diverge as $x_1 \to 0$ for every fixed $x_2 \in]0, 1[$. This condition is indeed fulfilled. Along the budget line, however,

$$\lim_{t \to 0} \Phi_{x_1}(t, 1 - t) = \lim_{t \to 0} \frac{1}{21}\left(t^{3/2} - t^2\right) = 0.$$

The simultaneous approach of the second commodity to its upper bound therefore changes the behavior of the first marginal utility. The prescribed coordinatewise divergence does not provide the endpoint behavior needed to obtain a change of sign of g. Proposition 3 thus deprives Amoroso's inductive strategy of its base, so that the induction attempted in Lemma IV cannot even start.

We conclude this section with Figure 5, which summarizes where the two-commodity AFT holds in the strictly positive quadrant of the (P, H)-plane. Only the region below the straight dashed line $H = 1 + P$ is relevant and it is partitioned according to the cases in Figures 1–4. Except for the area labeled Figure 1, the AFT fails in all other regions, because there always exists a $\Phi \in \mathcal{F}_+(2)$ defined over $]0, 1[^2$ such that the system of equations (3) has no solution in $]0, 1[^2$.

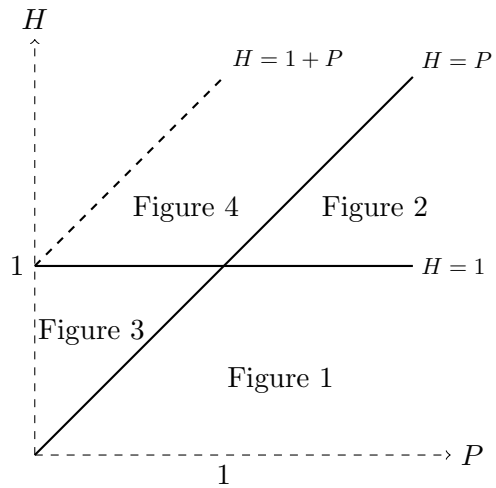


Figure 5: Space of parameters P and H and the validity of the AFT.

4 Amoroso's problem in the standard textbook formulation

In order to examine the scope of Amoroso's argument that the concept of consumer equilibrium must rest on a rigorous proof of its existence and, at the same time, to delineate clearly the limits of the results he obtains, it is convenient to formulate the problem he addresses in two variants that bring it closer to the formulations now regarded as standard and routinely presented in textbooks.

Accordingly, in this section we extend Amoroso's analysis in two directions. First, we drop assumption (A_2), allowing Φ to take non-positive values. Second, we replace the bounded consumption set $]0,1[^n$ with the strictly positive orthant, $D = \mathbb{R}^n_{++}$. The second extension retains the first, since positivity is not assumed in the unbounded setting either.

4.1 The role of the positivity assumption on the utility function

We begin with Amoroso's choice to restrict attention to utility functions that are everywhere positive. Two points are worth making: the assumption is a stable feature of Amoroso's own work, and it is inherited from Pareto. Amoroso states it explicitly in his 1921 lecture notes, where Φ denotes total ophelimity:

> *La ofelimità totale* misura allora il piacere derivante dal consumo simultaneo di date quantità di *tutte* le merci: se $x_1, x_2, \ldots, x_m$ sono queste quantità, si scrive:
>
> $$\Phi(x_1, x_2, \ldots, x_m)$$
>
> Φ essendo una funzione essenzialmente positiva. (Amoroso 1921, p. 66; emphasis in original)[15]

The same convention is adopted in his 1942 lecture notes, where indifference surfaces are indexed by positive constants:

> Consideriamo la funzione ofelimità $\Phi(x_1, x_2, \ldots, x_n)$ e prendiamo in esame quelle combinazioni $x_1, x_2, \ldots, x_n$, che — per essere ugualmente gradite al soggetto — attribuiscono alla Φ un medesimo valore. Esse verificano alla equazione
>
> $$\Phi(x_1, x_2, \ldots, x_n) = c,$$
>
> in cui c è una costante arbitraria positiva. (Amoroso 1942, p. 11)[16]

[15] "*Total ophelimity* then measures the pleasure derived from the simultaneous consumption of given quantities of *all* commodities: if $x_1, x_2, \ldots, x_m$ are these quantities, we write

$$\Phi(x_1, x_2, \ldots, x_m),$$

Φ being an essentially positive function." (Our translation).

[16] "We consider the ophelimity function $\Phi(x_1, x_2, \ldots, x_n)$ and examine those combinations $x_1, x_2, \ldots, x_n$ that—being equally agreeable to the individual—give Φ the same value. They satisfy the equation

$$\Phi(x_1, x_2, \ldots, x_n) = c,$$

where c is an arbitrary positive constant." (Our translation).

The 1942 passage does not state the assumption, but it presupposes it. Taken together, the two passages show that positivity brackets the 1928 article, spanning two decades of Amoroso's teaching, and is thus a stable feature of his framework. As anticipated, the assumption can be traced to Pareto (1906, ch. IV, §33): "l'ofelimità è sempre positiva; ed è questo un primo suo carattere."[17]

For the first extension, we reconsider Amoroso's Lemma III, adopting the same simplifications as before, but this time without assuming that Φ is positive on $D =]0,1[^2$. In view of Proposition 3, the only possibility is to consider parameters satisfying $0 < H < 1$ and $H < P$. Extending the approach used for Theorem 1 to the present setting is hindered by Example 1, which precludes relying on the boundedness of the marginal utility Φ_{x_1} when evaluating the limit (5). To amend Amoroso's argument, we address directly the indeterminate form $\infty - \infty$, which can arise in the limit

$$\lim_{s\to 0} g(s) = \lim_{s\to 0} P\Phi_{x_1}(x_s, s) - \lim_{s\to 0} \Phi_{x_2}(x_s, s).$$

Despite the mentioned difficulty, it is possible to resolve the indeterminacy by carefully exploiting the fact that the utility function satisfies assumptions (A_4) and (A_5). The proof of the next theorem, given in Appendix A.5, proceeds through three lemmas (Lemmas 1–3), which show that the result is less elementary than Amoroso's treatment might suggest.

Theorem 2 (Two-commodity AFT with utility of arbitrary sign)**.** *For two commodities, if $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_4), (A_5), and (A_6), $0 < H < 1$, and $H < P$, then the system of equations (3) has a unique solution.*

Comparing Theorems 1 and 2 clarifies the role of positivity in Amoroso's original treatment. Dropping (A_2) leaves unchanged the parameter region in which the AFT holds, but forces us to resolve the indeterminate form $\infty - \infty$ directly, by appealing to (A_4). Assumption (A_2) is thus dispensable for the validity of the result, but decisive for the simplicity of its proof. Theorem 2 also lays the groundwork for the second extension, to which we now turn: $D = \mathbb{R}^2_{++}$.

4.2 The unbounded consumption set

As before, commodity 1 is the numeraire and P is the price of commodity 2. Unlike in the previous subsection, we now assume that $D = \mathbb{R}^2_{++}$. Given a utility function $\Phi \in C^2(\mathbb{R}^2_{++})$ satisfying assumptions (A_4), (A_5), and (A_6), and given positive real numbers H and P, Amoroso's problem becomes that of proving the existence of $(x_1, x_2) \in \mathbb{R}^2_{++}$ that solves (3). Clearly, the budget line with $x_1, x_2 > 0$—the analogue of the segment $\overline{MN}$—is always entirely contained in the consumption set, since there are no upper bounds. Hence, no different cases arise. In effect, it is like having only the configuration depicted in Figure 1. The AFT on the unbounded consumption set with two commodities, Theorem 3 below, thus follows from Theorem 2 via a rescaling argument presented in Appendix A.6.

[17] "Ophelimity is always positive; this is a first characteristic of ophelimity." (Translation from Pareto 2014, p. 133).

Theorem 3 (Two-commodity AFT on $\mathbb{R}^2_{++}$)**.** *For two commodities, if* $\Phi \in C^2(\mathbb{R}^2_{++})$ *satisfies assumptions* (A_4)*,* (A_5)*, and* (A_6)*, and* $H, P > 0$*, then the system of equations (3) has a unique solution.*

Two important remarks are in order. First, assumption (A_7) plays no role in the unbounded-domain setting either. Second, there is a fundamental distinction between Theorems 2 and 3: the former guarantees the AFT only for some values of the parameters, as shown in Figure 5, while the latter places no restrictions on the validity of the AFT. This implies that the base (*i.e.*, $n = 2$) for Amoroso's inductive proof strategy, which is unavailable in the bounded setting, is now available. This suggests a conjecture about the general validity of the AFT for an arbitrary number n of commodities, provided the consumption set is unbounded. However, this is not the case, as the following example demonstrates (details in Appendix A.7).

Example 2 (Counterexample to the AFT on $\mathbb{R}^3_{++}$)**.** *Let* $D = \mathbb{R}^3_{++}$*,* $\mathbf{p} = (1, 1, P)$ *and*

$$\Phi(\mathbf{x}) = 3 + \sqrt{x_1 x_2} - e^{-x_1} - e^{-x_2} - e^{-x_3} + 2\sqrt{x_3}.$$

It is easily verified that $\Phi \in C^2(\mathbb{R}^3_{++})$ *is positive and satisfies assumptions* (A_4)*,* (A_5)*,* (A_6)*, and* (A_7)*. If the parameters* (P, H) *belong to the non-empty region*

$$\left\{ (P, H) \in \mathbb{R}^2_{++} : \ \frac{3P}{2} - \sqrt{\frac{P}{H}} - e^{-\frac{H}{P}} \leq 0 \right\},$$

then the system of equations

$$\begin{cases} p_2 \Phi_{x_1}(\mathbf{x}) - p_1 \Phi_{x_2}(\mathbf{x}) = 0 \\ p_3 \Phi_{x_1}(\mathbf{x}) - p_1 \Phi_{x_3}(\mathbf{x}) = 0 \\ \mathbf{p} \cdot \mathbf{x} = H \end{cases} \tag{6}$$

has no solution in $\mathbb{R}^3_{++}$*.*

Taken together, Proposition 3 and Example 2 show that the AFT does not hold in general. More specifically, in the two-commodity case, when the consumption set is bounded, as in Amoroso's original formulation, the theorem is guaranteed to hold only within a restricted region of the parameter space, contrary to the full generality claimed by Amoroso. With an unbounded consumption set, the validity of the AFT is limited by the number of commodities, since the theorem already fails with three commodities.

The move from two to three commodities has a simple economic interpretation. With two commodities, strictly positive prices and fixed positive wealth prevent both quantities from approaching zero simultaneously along the budget line. With three commodities, however, the first two quantities can approach zero simultaneously while expenditure is concentrated on the third. In the present example, this occurs along the feasible path

$$\mathbf{x}(t) = \left(t, t, \frac{H - 2t}{P} \right), \text{ with } 0 < t < \frac{H}{2}.$$

Although each of the first two marginal utilities diverges when its own quantity approaches zero with the others fixed, along this path

$$\lim_{t\to 0}\Phi_{x_1}(\mathbf{x}(t)) = \lim_{t\to 0}\Phi_{x_2}(\mathbf{x}(t)) = \lim_{t\to 0}\left(\frac{1}{2}+e^{-t}\right) = \frac{3}{2}.$$

Thus, the coordinatewise divergence required by (A_6) does not ensure the boundary behavior needed to extend the existence argument to more than two commodities.

5 The additive separability hypothesis

If the preceding sections can be seen as the *pars destruens* of the paper, this section is its *pars construens*.

A close inspection of our counterexamples reveals that they all involve utility functions that are not additively separable. This observation motivates an examination of whether, under additive separability:

– there exists a unique solution to the system of equations in (2);

– Amoroso's proof strategy, based on the intermediate value theorem and induction, can be completed.

These are two distinct issues, and we answer both in the affirmative. We note that the case of independent goods, which corresponds to additive separability, was well known to both Pareto and Amoroso.

> Parecchi degli autori che costituirono l'economia pura furono tratti, per rendere più trattabili i problemi che volevano studiare, ad ammettere che l'ofelimità di una merce dipendesse solo dalla quantità di quella merce a disposizione dell'individuo. (Pareto 1906, ch. IV, §11)[18]

Amoroso (1921, p. 68) and Amoroso (1942, p. 6) also consider the case of independent goods. However, Amoroso never invokes this case in the 1928 article, instead claiming full generality.

We start by stating the amended AFT both for Amoroso's original (bounded) setting and for our extension to the unbounded consumption set introduced in Section 4.

Theorem 4 (AFT with additive separability and a bounded D)**.** *For any number n of commodities, assume that $\Phi \in \mathcal{F}(n)$ is additively separable,* i.e.*,*

$$\Phi(\mathbf{x}) = \sum_{i=1}^{n} v_i(x_i),$$

so that $v_i \in C^2(]a_i, b_i[)$, $v_i' > 0$, $v_i'' < 0$, $\lim_{s\to a_i} v_i'(s) = +\infty$, and $\lim_{s\to b_i} v_i'(s) = 0$ for each $i = 1, \ldots, n$. Then, for every $\mathbf{p} \in \mathbb{R}^n_{++}$ and $H \in]\mathbf{p}\cdot\mathbf{a}, \mathbf{p}\cdot\mathbf{b}[$, the system of equations (2) *has a unique solution.*

[18] "A good many of the authors who developed pure economics were led, in an attempt to simplify the problems they wished to study, to assume that the ophelimity of a commodity depends only on the quantity of that commodity at an individual's disposal." (Translation from Pareto 2014, p. 127). In his annotations, reported in Pareto (2014, p. 127n4), John Chipman observes: "Here, by 'ophelimity of a commodity' Pareto clearly means what he calls its 'elementary ophelimity' (i.e., its marginal utility)."

Theorem 5 (AFT with additive separability and an unbounded D)**.** *For any number n of commodities, assume that $D = \mathbb{R}^n_{++}$ and that*

$$\Phi(\mathbf{x}) = \sum_{i=1}^{n} v_i(x_i),$$

with $v_i \in C^2(]0, +\infty[)$, $v_i' > 0$, $v_i'' < 0$, and $\lim_{s\to 0} v_i'(s) = +\infty$ for each $i = 1, \ldots, n$. Then, for every $\mathbf{p} \in \mathbb{R}^n_{++}$ and $H > 0$, the system of equations (2) has a unique solution.

Both theorems are proved in Appendix A.8 by means of an argument that does not use induction. Expressing the quantity of each commodity as a function of the Lagrange multiplier allows the intermediate value theorem to be applied directly to the resulting expenditure function. The two theorems settle the first issue but do not establish that Amoroso's own proof strategy can be completed. The second issue matters for our historical purpose, namely the comparison between the proof strategies of Amoroso (1928) and Wald (1935, 1936). In Appendix A.9, we present a proof that closely follows Amoroso's strategy, employing induction on the number of commodities and the intermediate value theorem. This addresses the second issue, showing that Amoroso's approach contains a workable method once the class of utility functions is restricted and the missing arguments are supplied. Indeed, since Amoroso overlooked some cases (Figures 2–4), both the base for the induction and the inductive step require arguments for instances he did not consider. Supplying these arguments makes the role of assumption (A_7) transparent in the bounded case and confirms that it is dispensable in the unbounded case.[19] At the same time, one cannot help noting that, once the details are made explicit, Amoroso's proof strategy yields a considerably longer proof than the one presented in Appendix A.8.

Theorem 4, albeit of more limited scope than the original "Fundamental Theorem" (1928), undoubtedly constitutes a rigorous answer to the question that Amoroso set out to address in his work: establishing a valid existence foundation for the first-order characterization of consumer equilibrium.

6 A comparison with Wald (1936)

In March and November 1934, Wald presented to Karl Menger's Mathematical Colloquium in Vienna two contributions on the existence of a competitive equilibrium. They appeared in the issues of the *Ergebnisse eines mathematischen Kolloquiums* for 1933–34 and 1934–35, which were printed in 1935 and 1936, respectively. These contributions were examined in detail by Mutoh (2003) and Hildenbrand

[19] A meticulous reading of Amoroso (1928) shows that the first and only use of assumption (A_7) is in Lemma IV (p. 40), at the endpoint he labels P, where $x_n = b_n$: "Per R coincidente con P, invece, sempre in forza delle (7), si ha che la derivata $\frac{\partial \Phi}{\partial x_n}$ è nulla." ("When R coincides with P, on the other hand, then on the basis of (7) the derivative $\frac{\partial \Phi}{\partial x_n}$ equals zero.") The use presupposes that, with reference to Amoroso's notation, $h_n - p_n b_n$ belongs to the admissible range for the $(n-1)$-commodity system, which does not follow from his formula (16) on p. 39, and in general fails.

(1998), respectively. As pointed out by Weintraub (1983), the main difference between the two concerns the generality of the assumptions; as these reconstructions show, both proofs rely on the intermediate value theorem for correspondences and induction on the number of commodities. Wald's proofs therefore combine the same two methodological elements—an intermediate value argument and induction—that Amoroso had employed six years before Wald's presentations, in his attempted existence proof. The observation by Guccione and Minelli (1999) concerning the precursory character of Amoroso's axiomatic approach to economic theory can thus be supplemented by a more specific comparison of his proof strategy with Wald's. As we see below, what separates the two arguments is not the strategy itself but the formulation adopted for the boundary condition.

John (1999) examines which of Wald's assumptions support his inductive method. His focus is Wald's assumption 6, condition (W_6) below, which is equivalent to the weak axiom of revealed preference for market demand. Against the view of Dorfman et al. (1958, p. 368) that this assumption serves only to establish uniqueness, he shows that it is crucial to the existence part of Wald's induction proof: it guarantees the uniqueness of equilibrium in every lower-dimensional subsystem, and it is precisely this property that enables the inductive step. His reformulation of Wald's proof further shows that convexity of the solution sets of these subsystems, rather than their uniqueness, suffices for an inductive proof of this kind. A comparison of proof strategies must therefore examine the assumptions that make the induction work, rather than merely identify a shared sequence of mathematical steps. From this perspective, the counterpart of Wald's assumption 6 is available in Amoroso's setting, as we show below. It is Amoroso's coordinatewise boundary condition (A_6) that does not imply the boundary condition (W_5) required by Wald.

The economic problems addressed by Amoroso and Wald are not immediately comparable. Amoroso studied consumer equilibrium at given prices and wealth, whereas Wald considered competitive equilibrium in an economy with production, where commodity and factor prices must be compatible with demand, technology, and resource availability. Their primitives also differ: Amoroso starts from a utility function, while Wald makes assumptions directly on an inverse demand function. The comparison below identifies a specialization in which these two formulations can be connected through an economic reinterpretation of Amoroso's setting. Furthermore, the comparison concerns our extension of Amoroso's analysis to the strictly positive orthant, rather than his original bounded consumption set.

Let us consider the more general of Wald's two problems, that of Wald (1936), as presented by Hildenbrand (1998). There are n consumption commodities, with quantity vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$, and m factors of production. The production matrix A has m rows and n columns; its entry a_{ij} gives the quantity of factor i required to produce one unit of commodity j. Given an inverse demand function $\mathbf{f} : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ and a factor-endowment vector $\mathbf{r}$, Wald defines an equilibrium as a solution $(\mathbf{x}, \boldsymbol{\pi}, \mathbf{q}, \mathbf{u})$ of the following system of equations:

$$\begin{aligned} &\boldsymbol{\pi} = \mathbf{f}(\mathbf{x}), \\ &\mathbf{r} = A\mathbf{x} + \mathbf{u}, \\ &\boldsymbol{\pi}^T = \mathbf{q}^T A, \\ &q_i u_i = 0, \ i = 1, \ldots, m \end{aligned} \tag{7}$$

where $\boldsymbol{\pi} \in \mathbb{R}^n_{++}$ is the vector of commodity prices, $\mathbf{q} \in \mathbb{R}^m_+$ is the vector of factor prices, and $\mathbf{u} \in \mathbb{R}^m_+$ can be interpreted as the vector of unused (excess) factor supply.

Wald shows existence of equilibria under the following six assumptions:

(W_1) The factor-endowment vector $\mathbf{r}$ belongs to $\mathbb{R}^m_{++}$.

(W_2) All entries a_{ij} of the production matrix are non-negative.

(W_3) No column of the production matrix is zero.

(W_4) The inverse demand function is continuous.

(W_5) If a sequence $(\mathbf{x}^\nu)_{\nu\in\mathbb{N}}$ with $\mathbf{x}^\nu \in \mathbb{R}^n_{++}$ converges to $\mathbf{x} \in \mathbb{R}^n_+$, then $f_j(\mathbf{x}^\nu)$ tends to infinity if $x_j = 0$ and remains bounded if $x_j > 0$.

(W_6) The inverse demand function satisfies $\mathbf{f}(\mathbf{x}) \cdot (\mathbf{x}' - \mathbf{x}) \leq 0 \Rightarrow \mathbf{f}(\mathbf{x}') \cdot (\mathbf{x}' - \mathbf{x}) < 0$, whenever $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^n_{++}$ with $\mathbf{x} \neq \mathbf{x}'$.

He also establishes uniqueness of commodity quantities, commodity prices, and unused factor supplies, while factor prices need not be unique in the general model.

Now, consider an economy with a single agent and a single production factor (à la Robinson Crusoe). The agent is at once consumer, owner of the labor endowment, and producer. The production matrix reduces to a strictly positive vector $\mathbf{a} = (a_1, \ldots, a_n)$, where a_i is the labor required to produce one unit of commodity i. The system of equations (7) in this case can be equivalently written as

$$\begin{aligned} &\mathbf{f}(\mathbf{x}) = q\mathbf{a}, \\ &r = \mathbf{a} \cdot \mathbf{x} + u, \\ &qu = 0. \end{aligned} \tag{8}$$

Setting $r = H$, $\mathbf{a} = \mathbf{p}$, and $\mathbf{f} = \nabla\Phi$, and assuming (A_1), (A_3), and (A_4) on $D = \mathbb{R}^n_{++}$, system (8) reduces to Amoroso's system (2), and (W_1)–(W_4) hold.[20]

This identification gives Amoroso's coefficients a different economic interpretation: the given commodity prices become labor requirements per unit of output, while wealth becomes the available labor endowment. The constraint therefore describes the technological possibilities of an economy in which consumption must be supplied by the agent's own production. Its equality form expresses full employment of the available labor.

The existence of a consumer equilibrium can be proved by appealing to Wald's theorem (see Hildenbrand 1998, p. 55) whenever (W_5) and (W_6) hold. Amoroso's assumptions, however, do not guarantee this possibility, as Example 2 shows. By contrast, under additive separability, (W_5) and (W_6) hold. Let $(\mathbf{x}^\nu)_{\nu\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n_{++}$ converging to $\mathbf{x} \in \mathbb{R}^n_+$. If $x_j = 0$, then $f_j(\mathbf{x}^\nu) = v_j'(x_j^\nu) \to \infty$, since, under separability, the boundary condition (A_6) reads $\lim_{s\to 0} v_j'(s) = \infty$;[21]

[20] Since marginal utilities and unit labor requirements are strictly positive, the first equation in (8) implies $q > 0$. The complementarity condition therefore gives $u = 0$, and eliminating q yields Amoroso's system of equations (2).

[21] In this form, assumption (A_6) is close in spirit to Condition 4 in Wald (1935, p. 12), where the inverse demand for each commodity is assumed to depend only on the quantity of that commodity, as Weintraub (1983, p. 9) notes when referring to "the simplification introduced by assuming that the demand price for good j is a function only of the quantity of good j."

if $x_j > 0$, then $f_j(\mathbf{x}^\nu) \to v_j'(x_j)$ by the continuity of v_j', so that $f_j(\mathbf{x}^\nu)$ remains bounded. Hence, (W_5) holds. Next, from $\mathbf{f}(\mathbf{x}) = (v_1'(x_1), \ldots, v_n'(x_n))$, we have

$$\big(\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x})\big) \cdot (\mathbf{x}' - \mathbf{x}) = \sum_{i=1}^{n} \big(v_i'(x_i') - v_i'(x_i)\big)(x_i' - x_i) < 0.$$

Indeed, each v_i' is strictly decreasing, so that every summand is non-positive and at least one of them is negative, as $\mathbf{x} \neq \mathbf{x}'$. Therefore, (W_6) follows. It is worth noting that additive separability is not needed for (W_6). Indeed, assumption (A_5) implies strict concavity of Φ on the convex domain D, and hence

$$\big(\nabla\Phi(\mathbf{x}') - \nabla\Phi(\mathbf{x})\big) \cdot (\mathbf{x}' - \mathbf{x}) < 0 \quad \text{whenever } \mathbf{x}' \neq \mathbf{x}.$$

In other words, although Amoroso's system can be viewed as a special case of Wald's, assumption (A_6) is weaker than (W_5),[22] and this prevents a general application of Wald's existence theorem under Amoroso's assumptions. Under separability, however, this obstruction disappears: when each marginal utility depends only on its own quantity, the coordinatewise condition (A_6) controls every approach to the boundary.

At a solution $\mathbf{x}^*$ of (2), the equations

$$\pi_i = \Phi_{x_i}(\mathbf{x}^*) = qa_i, \quad \text{for each} \quad i = 1, \ldots, n,$$

equate the marginal valuation of each commodity with the value of the labor required to produce it: the multiplier q is the shadow value of labor and π_i is the corresponding commodity shadow price, both expressed using the chosen utility representation. These shadow prices support an interpretation in which consumption and production decisions are separated. As the owner of the labor endowment, the agent receives income qH. At commodity prices $\boldsymbol{\pi} = q\mathbf{a}$, his budget constraint $\boldsymbol{\pi} \cdot \mathbf{x} \leq qH$ coincides with the technological constraint $\mathbf{a} \cdot \mathbf{x} \leq H$. Meanwhile, each production activity, which exhibits constant returns to scale, earns zero profits because its commodity price equals its unit labor cost. The allocation therefore satisfies consumer optimality, producer optimality, and full employment of the available factor.

The economic scope of this identification is limited. With a single factor and fixed unit labor requirements, relative commodity prices are determined by technology, while preferences determine the equilibrium allocation of labor and consumption. Moreover, the presence of a single consumer removes the problem of aggregating individual demands.[23] Within these restrictions, Amoroso's consumer-equilibrium equations can therefore be interpreted as the equilibrium conditions of a special case of Wald's production economy.

[22] Recall that the utility function of Example 2 does not satisfy (W_5) since along the path $\mathbf{x}(t) = (t, t, (H - 2t)/P)$, the derivative $\Phi_{x_1} = f_1$ remains bounded as $t \to 0$, although the limit of $\mathbf{x}(t)$ has first coordinate zero.

[23] Dorfman et al. (1958, p. 368) observe that, in imposing the weak axiom, "Wald really assumed that there is essentially only one rational consumer"; the single-consumer restriction is therefore less severe than it may appear.

7 Conclusions

Our analysis gives a more precise meaning to the claim that Amoroso anticipated Wald. Besides its explicitly axiomatic formulation, Amoroso's contribution contains two principal components of Wald's subsequent proof strategy: an intermediate value argument and induction on the number of commodities. Yet this anticipation must be distinguished from a successful proof of the general result announced in 1928. We supply missing arguments in the case considered by Amoroso, construct counterexamples to the broader claim, and show that the proposed strategy can be completed under additive separability.

These findings qualify the earlier assessments of Guerraggio (1990) and Legris (1995). The mathematical analysis refines the historical attribution by distinguishing a valid restricted result, an unsuccessful general claim, and a proof strategy that becomes viable under additional assumptions. The comparison with Wald likewise separates two questions: whether Amoroso's equations are a special case of Wald's, which they are, and whether Amoroso's assumptions imply Wald's, which they do not without additive separability. In particular, it is Amoroso's coordinatewise boundary condition that does not imply Wald's boundary condition. Mathematical precision is therefore essential here to determining both the achievement and the limitations of Amoroso's contribution.

We conclude by observing that Wald very likely developed his proof strategy independently: Amoroso (1928) remained an isolated contribution that Amoroso himself never developed further. As Pomini (2022, p. 53) notes: "In the years that immediately followed, Amoroso did not continue along this path of research but changed direction, dedicating himself to dynamics." Moreover, in Amoroso (1942)—a collection of his lectures delivered at the *Istituto Nazionale di Alta Matematica* (INdAM) in Rome in 1940–1941—when discussing the problem of the existence of solutions (p. 23), he referred only to his own 1928 contribution, without amending it or citing any of Wald's work. It would also be worth investigating whether further evidence links the works of Amoroso and Wald.

A Appendix: Proofs

A.1 Proof of Proposition 1

Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions defined by

$$f_n(x_1) := \Phi_{x_2}\left(x_1, \frac{1}{n}\right),$$

for each $x_1 \in]0,1[$. Fix an $s_0 \in]0, \min\{1, \frac{H}{P}\}[$ and let $x_{s_0} = H - Ps_0 = H - \delta$. Clearly, for each $s \in]0, s_0]$, it follows that x_s belongs to the compact interval $K := [H-\delta, H] \subset]0,1[$. For each n sufficiently large, say $n \geq n_0$ with $\frac{1}{n_0} < s_0$, the functions $f_n(x_1)$ are real-valued, continuous, and well defined on the compact set K, *i.e.*,

$$f_n : [H-\delta, H] \to \mathbb{R}.$$

Furthermore, assumption (A_5) implies that the partial derivative $\Phi_{x_2}(x_1, \cdot)$ is a strictly decreasing function for every $x_1 \in]0,1[$. But then, the sequence $(f_n)_{n\geq n_0}$ is

increasing,

$$f_n(x_1) = \Phi_{x_2}\left(x_1, \frac{1}{n}\right) < \Phi_{x_2}\left(x_1, \frac{1}{n+1}\right) = f_{n+1}(x_1), \text{ for each } x_1 \in K.$$

Moreover, assumption (A_6) implies that the sequence $(f_n)_{n\geq n_0}$ diverges pointwise on K, *i.e.*,

$$\lim_{n\to\infty} \Phi_{x_2}\left(x_1, \frac{1}{n}\right) = \lim_{x_2\to 0} \Phi_{x_2}(x_1, x_2) = \infty, \text{ for each } x_1 \in K.$$

By applying a slight extension of Dini's theorem, as we consider a monotone sequence which diverges, we conclude that the sequence $(f_n)_{n\geq n_0}$ diverges uniformly to ∞ on K. This means that for each $M > 0$ there exists an $\underline{n} \geq n_0$ such that $n \geq \underline{n}$ implies that

$$f_n(x_1) > M, \text{ for each } x_1 \in K.$$

Now, consider an s such that $0 < s < \frac{1}{\underline{n}}$. Then, $x_s = H - Ps \in K$. Since $\Phi_{x_2}(x_s, \cdot)$ is strictly decreasing, it follows that

$$\Phi_{x_2}(x_s, s) > \Phi_{x_2}\left(x_s, \frac{1}{\underline{n}}\right) = f_{\underline{n}}(x_s) > M.$$

Finally, since $M > 0$ is arbitrary, it follows that

$$\lim_{s\to 0} \Phi_{x_2}(x_s, s) = \infty.$$

□

A.2 Proof of Example 1

It is immediate to verify that assumptions (A_0), (A_3), (A_4), (A_6), and (A_7) are satisfied. We now show that (A_5) is also satisfied. We first observe that

$$\Phi_{x_1x_1} = -\frac{1}{2}x_1^{-\frac{3}{2}} - 2(x_2^{-1} + x_2) < 0,$$

for each $(x_1, x_2) \in]0, 1[^2$. Next, we study the sign of the determinant of the Hessian matrix given by $\Phi_{x_1x_1}\Phi_{x_2x_2} - (\Phi_{x_1x_2})^2$. We also observe that $\Phi_{x_1x_1}\Phi_{x_2x_2} = A + B + C + D$ with

$$A = \frac{1}{4}(x_1x_2)^{-\frac{3}{2}},\ B = x_1^{-\frac{3}{2}}(1-x_1)^2x_2^{-3},\ C = x_2^{-\frac{5}{2}} + x_2^{-\frac{1}{2}},\ D = 4(1+x_2^2)(1-x_1)^2x_2^{-4}.$$

Since A, B, and C are positive and $1 + x_2^2 > 1$, the following inequality holds

$$\Phi_{x_1x_1}\Phi_{x_2x_2} > D > 4(1-x_1)^2x_2^{-4},$$

for each $(x_1, x_2) \in]0, 1[^2$. But then,

$$\Phi_{x_1x_1}\Phi_{x_2x_2} - (\Phi_{x_1x_2})^2 > 4(1-x_1)^2x_2^{-4} - 4(1-x_1)^2\left(1 - x_2^{-2}\right)^2 =$$
$$4(1-x_1)^2\left(x_2^{-4} - \left(1 - x_2^{-2}\right)^2\right) = 4(1-x_1)^2(2x_2^{-2} - 1) > 0,$$

where the last inequality follows from the fact that $x_2 \in]0, 1[$. Hence, the Hessian matrix is negative definite on $]0, 1[^2$ and assumption (A_5) is also satisfied. We then conclude that $\Phi \in \mathcal{F}(2)$. □

A.3 Proof of Proposition 2

Let us fix a point (x_{s_1}, s_1) where $0 < s_1 < \min\{1, \frac{H}{P}\}$. We then consider s and s_0 such that $0 < s \leq s_0 < s_1$ as illustrated in Figure 6, which depicts the particular case $H < P$.

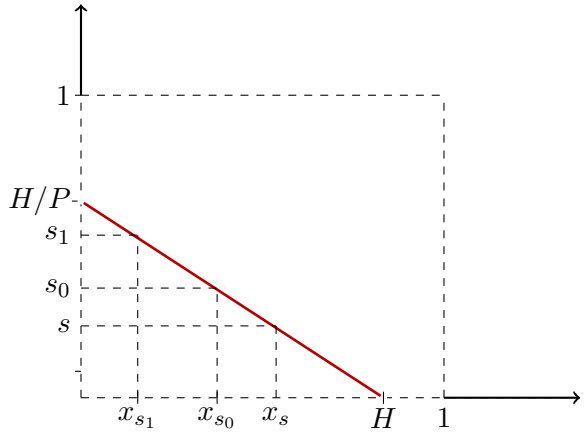


Figure 6

Since Φ is concave by assumption (A_5), it follows that

$$\Phi(x_{s_1}, s) \leq \Phi(x_s, s) + \Phi_{x_1}(x_s, s)(x_{s_1} - x_s)$$

that can be equivalently written as

$$\Phi_{x_1}(x_s, s) \leq \frac{\Phi(x_s, s) - \Phi(x_{s_1}, s)}{x_s - x_{s_1}}.$$

Since Φ is concave and D is bounded, $\Phi(x_1, x_2) \leq L$ for an $L > 0$, for each $(x_1, x_2) \in]0, 1[^2$. Hence, since $\Phi > 0$, we conclude that

$$\Phi_{x_1}(x_s, s) \leq \frac{|\Phi(x_s, s) - \Phi(x_{s_1}, s)|}{x_s - x_{s_1}} \leq \frac{|\Phi(x_s, s)| + |\Phi(x_{s_1}, s)|}{x_s - x_{s_1}} \leq \frac{\Phi(x_s, s) + \Phi(x_{s_1}, s)}{x_s - x_{s_1}} \leq \frac{2L}{x_s - x_{s_1}} \leq \frac{2L}{x_{s_0} - x_{s_1}},$$

for each $s \in]0, s_0]$. That is, $\Phi_{x_1}(x_s, s)$ is bounded above on every interval $]0, s_0[$ with $s_0 < \min\{\frac{H}{P}, 1\}$. □

A.4 Proof of Proposition 3

Under $H < 1 + P$, we have to consider two cases: either $P \leq H$ or $P > H \geq 1$. We start by considering $P \leq H$ which covers two subcases:

- $P \leq H < 1$, corresponding to Figure 3, where the domain of g is the whole $]0, 1[$
- $\max\{1, P\} \leq H < 1 + P$, corresponding to Figure 4, where the domain of g is $]\frac{H-1}{P}, 1[\subset]0, 1[$.

Consider the function

$$\Phi(x_1, x_2) = \varepsilon(1 - x_2)^2 f(x_1) + f(x_2)$$

defined on $D =]0, 1[^2$, where $f(x) = 2\sqrt{x} - x$ and $0 < \varepsilon < \min\left(\frac{1}{20}, \frac{1}{2(2+\sqrt{P})}\right)$. For every $x \in]0, 1[$, the following facts hold for f:

(i) $0 < f(x) < 1$;

(ii) $f'(x) = x^{-\frac{1}{2}} - 1 = \frac{1-x}{\sqrt{x}(1+\sqrt{x})} > \frac{1-x}{2}$;

(iii) $(f'(x))^2 < \frac{1}{x}$;

(iv) $f''(x)f''(y) > \frac{1/4}{x^{3/2}}$.

It is immediate to verify that assumptions (A_0), (A_2), and (A_3) are satisfied. It is also straightforward to see that

$$\Phi_{x_1}(x_1, x_2) = \varepsilon(1 - x_2)^2 f'(x_1) > 0.$$

Furthermore,

$$\Phi_{x_2}(x_1, x_2) = -2\varepsilon(1 - x_2)f(x_1) + f'(x_2) > -2\varepsilon(1 - x_2)f(x_1) + \frac{1 - x_2}{2} = (1 - x_2)\left(\frac{1}{2} - 2\varepsilon f(x_1)\right) > (1 - x_2)\left(\frac{1}{2} - 2\varepsilon\right) > 0,$$

where the first inequality follows from (ii), the second inequality follows from (i), and the last one follows from the fact that $\varepsilon < 1/4$. Hence, assumption (A_4) is satisfied. Next, we observe that

$$\Phi_{x_1x_1}(x_1, x_2) = \varepsilon(1 - x_2)^2 f''(x_1) = -\varepsilon(1 - x_2)^2 \frac{1}{2} x_1^{-\frac{3}{2}} < 0$$

and that the determinant of the Hessian matrix is

$$\Phi_{x_1x_1}\Phi_{x_2x_2} - (\Phi_{x_1x_2})^2 = \varepsilon(1 - x_2)^2 B(x_1, x_2)$$

with $B(x_1, x_2) = f''(x_1)f''(x_2) + 2\varepsilon f''(x_1)f(x_1) - 4\varepsilon(f'(x_1))^2$. By (i), (iii), and (iv), it follows that

$$B(x_1, x_2) > f''(x_1)f''(x_2) + 2\varepsilon f''(x_1) - \frac{4\varepsilon}{x_1} > \frac{1/4 - \varepsilon}{x_1^{3/2}} - \frac{4\varepsilon}{x_1} > \frac{1/4 - 5\varepsilon}{x_1^{3/2}}$$

with the last inequality following from the fact that $\frac{4\varepsilon}{x_1} = \frac{4\varepsilon\sqrt{x_1}}{x_1^{3/2}} < \frac{4\varepsilon}{x_1^{3/2}}$, for $0 < x_1 < 1$. Therefore,

$$\Phi_{x_1x_1}\Phi_{x_2x_2} - (\Phi_{x_1x_2})^2 > \varepsilon(1 - x_2)^2 \frac{1/4 - 5\varepsilon}{x_1^{3/2}} > 0,$$

as $\varepsilon < 1/20$. Hence, assumption (A_5) is satisfied. It is immediate to verify that assumptions (A_6) and (A_7) are also satisfied and we then conclude that $\Phi \in \mathcal{F}_+(2)$.

Next, we observe that

$$f'(x_s) = x_s^{-\frac{1}{2}} - 1 = (H - Ps)^{-\frac{1}{2}} - 1 < (P(1-s))^{-\frac{1}{2}}, \tag{9}$$

where the last inequality follows from the fact that $H \geq P$. We then conclude that

$$\begin{aligned} g(s) = P\varepsilon(1-s)^2 f'(x_s) + 2\varepsilon(1-s)f(x_s) - f'(s) < \\ P\varepsilon(1-s)^2(P(1-s))^{-\frac{1}{2}} + 2\varepsilon(1-s) + \frac{s-1}{2} = \\ \sqrt{P}\varepsilon(1-s)^{\frac{3}{2}} + 2\varepsilon(1-s) + \frac{s-1}{2} < \\ (1-s)\left(\varepsilon\sqrt{P} + 2\varepsilon - \frac{1}{2}\right) < 0, \end{aligned}$$

where the first inequality follows by (9), by (i), and by (ii), the second inequality follows from the fact that either $s \in]0,1[$ or $s \in]\frac{H-1}{P}, 1[$, and the third inequality follows from the fact that $\varepsilon < \frac{1}{2(2+\sqrt{P})}$. Since $g(s) < 0$ on its domain, no solution exists to the system of equations (3), *i.e.*, the AFT fails.

It remains to consider the case $P > H \geq 1$ corresponding to Figure 2, where the domain of g is $]\frac{H-1}{P}, \frac{H}{P}[\subset]0,1[$. We observe that such a case is symmetric to Figure 3 by swapping the two variables. So for the function

$$\Phi(x_1, x_2) = \varepsilon(1-x_1)^2 f(x_2) + f(x_1),$$

it is possible to show, by following *mutatis mutandis* the same steps, that $g(s)$ is positive on its domain. Then, no solution exists to the system of equations (3), *i.e.*, the AFT fails. □

A.5 Proof of Theorem 2

The theorem can be proved by first establishing the following three lemmas.

Lemma 1. *Assume that $H < 1$ and that $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_4), (A_5), and (A_6). Then,*

$$\lim_{s \to 0} \frac{\Phi_{x_1}(x_s, s)}{\Phi_{x_2}(x_s, s)} = 0.$$

Proof. As in the proof of Proposition 2, let us fix a point (x_{s_1}, s_1) where $0 < s_1 < \min\{1, \frac{H}{P}\}$. We then consider s and s_0 such that $0 < s \leq s_0 < s_1$ as illustrated in Figure 6. Since Φ is concave by assumption (A_5), it follows that

$$\Phi(x_{s_1}, s_0) \leq \Phi(x_s, s) + \nabla\Phi(x_s, s) \cdot \left(x_{s_1} - x_s, s_0 - s\right),$$

which can be written as

$$\Phi_{x_1}(x_s, s)(x_s - x_{s_1}) \leq \Phi(x_s, s) - \Phi(x_{s_1}, s_0) + \Phi_{x_2}(x_s, s)(s_0 - s).$$

By dividing the previous inequality by $(x_s - x_{s_1})\Phi_{x_2}(x_s, s)$, which is positive, we obtain

$$\frac{\Phi_{x_1}(x_s, s)}{\Phi_{x_2}(x_s, s)} \leq \frac{\Phi(x_s, s) - \Phi(x_{s_1}, s_0)}{(x_s - x_{s_1})\,\Phi_{x_2}(x_s, s)} + \frac{s_0 - s}{x_s - x_{s_1}}. \tag{10}$$

We next observe that the concavity of Φ also implies that

$$\Phi(x_s, s_0) \leq \Phi(x_s, s) + \Phi_{x_2}(x_s, s)(s_0 - s)$$

and that $\Phi(x_1, x_2) \leq L$, for an $L > 0$, for each $(x_1, x_2) \in]0, 1[^2$. But then,

$$\Phi(x_s, s_0) - \Phi_{x_2}(x_s, s)(s_0 - s) \leq \Phi(x_s, s) \leq L.$$

Therefore, by applying the elementary fact that $a \leq x \leq b$ implies $|x| \leq \max\{|a|, |b|\} \leq |a| + |b|$, we obtain the following result

$$|\Phi(x_s, s)| \leq L + |\Phi(x_s, s_0)| + \Phi_{x_2}(x_s, s)(s_0 - s).$$

By dividing the previous inequality by $\Phi_{x_2}(x_s, s)$, which is positive, we obtain

$$\frac{|\Phi(x_s, s)|}{\Phi_{x_2}(x_s, s)} \leq \frac{L}{\Phi_{x_2}(x_s, s)} + \frac{|\Phi(x_s, s_0)|}{\Phi_{x_2}(x_s, s)} + (s_0 - s).$$

By Proposition 1, it follows straightforwardly that

$$\limsup_{s \to 0} \frac{|\Phi(x_s, s)|}{\Phi_{x_2}(x_s, s)} \leq s_0. \tag{11}$$

Since $\frac{|\Phi(x_s,s)|}{\Phi_{x_2}(x_s,s)} \geq 0$ by assumption (A_4) and since inequality (11) holds for each $s_0 \in]0, s_1[$, it follows that

$$\lim_{s \to 0} \frac{|\Phi(x_s, s)|}{\Phi_{x_2}(x_s, s)} = 0. \tag{12}$$

Given this last result, we consider the inequality (10) and we observe that the terms of the right-hand side are such that

$$\lim_{s \to 0} \frac{\Phi(x_s, s)}{(x_s - x_{s_1})\Phi_{x_2}(x_s, s)} = \frac{0}{H - x_{s_1}} = 0,$$

by the limit in (12),

$$\lim_{s \to 0} \frac{\Phi(x_{s_1}, s_0)}{\Phi_{x_2}(x_s, s)(x_s - x_{s_1})} = \frac{\Phi(x_{s_1}, s_0)}{\infty(H - x_{s_1})} = 0,$$

by Proposition 1, and

$$\lim_{s \to 0} \frac{s_0 - s}{x_s - x_{s_1}} = \frac{s_0}{H - x_{s_1}}.$$

But then,

$$\limsup_{s \to 0} \frac{\Phi_{x_1}(x_s, s)}{\Phi_{x_2}(x_s, s)} \leq \frac{s_0}{H - x_{s_1}}.$$

Since $\frac{\Phi_{x_1}(x_s,s)}{\Phi_{x_2}(x_s,s)} > 0$ by assumption (A_4) and since $s_0 > 0$ is arbitrary, we conclude that

$$\lim_{s \to 0} \frac{\Phi_{x_1}(x_s, s)}{\Phi_{x_2}(x_s, s)} = 0.$$

□

Lemma 2. *Assume that $H < 1$ and that $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_4), (A_5), and (A_6). Then,*

$$\lim_{s\to 0} g(s) = -\infty. \tag{13}$$

Proof. Since the function $g(s)$ can be written as

$$g(s) = \Phi_{x_2}(x_s, s)\left(P\frac{\Phi_{x_1}(x_s,s)}{\Phi_{x_2}(x_s,s)} - 1\right),$$

it follows immediately that

$$\lim_{s\to 0} g(s) = \lim_{s\to 0} \Phi_{x_2}(x_s,s)\left(P\frac{\Phi_{x_1}(x_s,s)}{\Phi_{x_2}(x_s,s)} - 1\right) = \infty(0-1) = -\infty,$$

by Proposition 1 and Lemma 1. □

Having determined the behavior of the function g close to point N, we now need to analyze the behavior around M. Since both $D =]0,1[^2$ and the class of functions $\mathcal{F}(2)$ are symmetric under the interchange of variables x_1 and x_2, we can determine the behavior of g near M directly from Lemma 2. It is then immediate to obtain the following lemma.

Lemma 3. *Assume that $H < P$ and that $\Phi \in C^2(]0,1[^2)$ satisfies assumptions (A_4), (A_5), and (A_6). Then*

$$\lim_{s\to \frac{H}{P}} g(s) = \infty.$$

Proof. We first make explicit how the definition of g depends on Φ by writing

$$g_\Phi(s) = P\Phi_{x_1}(H-Ps,s) - \Phi_{x_2}(H-Ps,s), \text{ for each } s \in \left]\max\{0, \frac{H-1}{P}\}, H/P\right[.$$

We next define $\Psi(x_1,x_2) := \Phi(x_2,x_1)$, $\bar{H} = \frac{H}{P}$, and $\bar{P} = \frac{1}{P}$, and consider

$$g_\Psi(t) = \bar{P}\,\Psi_{x_1}(\bar{H}-\bar{P}t,t) - \Psi_{x_2}(\bar{H}-\bar{P}t,t), \text{ for each } t \in]0, \min\{1,H\}[.$$

Since $D =]0,1[^2$ and the class of functions $\mathcal{F}(2)$ are symmetric under the interchange of variables x_1 and x_2, it follows that $\Psi \in \mathcal{F}(2)$ whenever $\Phi \in \mathcal{F}(2)$. Moreover, since $H < P$ by hypothesis, we have $\bar{H} = \frac{H}{P} < 1$, so Lemma 2 applies to Ψ and yields

$$\lim_{t\to 0} g_\Psi(t) = -\infty.$$

We now relate g_Ψ to g_Φ. By the chain rule, $\Psi_{x_1}(x_1,x_2) = \Phi_{x_2}(x_2,x_1)$ and $\Psi_{x_2}(x_1,x_2) = \Phi_{x_1}(x_2,x_1)$, so that

$$g_\Psi(t) = \frac{1}{P}\,\Phi_{x_2}\left(t, \frac{H-t}{P}\right) - \Phi_{x_1}\left(t, \frac{H-t}{P}\right).$$

Setting $s = \dfrac{H-t}{P}$, we obtain

$$g_\Phi(s) = P\Phi_{x_1}(H-Ps,s) - \Phi_{x_2}(H-Ps,s) = P\Phi_{x_1}(t,s) - \Phi_{x_2}(t,s) = -P\,g_\Psi(t).$$

Since $t \to 0$ is equivalent to $s \to \frac{H}{P}$, it follows that

$$\lim_{s \to \frac{H}{P}} g_\Phi(s) = -P \lim_{t \to 0} g_\Psi(t) = -P \cdot (-\infty) = \infty.$$

□

Given the limits established in Lemmas 2 and 3, we can conclude, by a straightforward application of the intermediate value theorem, that the AFT holds in the two-commodity case with utility of arbitrary sign, for the parameter configuration of Figure 1.

A.6 Proof of Theorem 3

We start by observing that only the restriction of Φ to the bounded rectangle

$$]0, H[\times \left]0, \frac{H}{P}\right[$$

matters. We fix $A > \max\left\{H, \frac{H}{P}\right\}$, and set $\bar{H} := \frac{H}{A}$. Then, it is immediate to note that

$$0 < \bar{H} < 1 \quad \text{and} \quad \frac{\bar{H}}{P} = \frac{H}{AP} < 1 \implies \bar{H} < P,$$

i.e., the parameters $(P, \bar{H})$ belong to the region, shown in Figure 5, where the AFT is true. Since the function

$$\Psi(y_1, y_2) := \Phi(Ay_1, Ay_2), \text{ for each } (y_1, y_2) \in]0, 1[^2,$$

satisfies the assumptions (A_4), (A_5), and (A_6), we apply Theorem 2 to conclude that there exists (y_1^*, y_2^*) such that

$$P\Psi_{y_1}(y_1^*, y_2^*) - \Psi_{y_2}(y_1^*, y_2^*) = 0 \quad \text{and} \quad y_1^* + Py_2^* = \bar{H}. \tag{14}$$

We define $x_1^* = Ay_1^*$ and $x_2^* = Ay_2^*$. Since $\Psi_{y_1}(y_1^*, y_2^*) = A\Phi_{x_1}(Ay_1^*, Ay_2^*)$ and $\Psi_{y_2}(y_1^*, y_2^*) = A\Phi_{x_2}(Ay_1^*, Ay_2^*)$, the equations in (14) can be written as

$$AP\Phi_{x_1}(x_1^*, x_2^*) - A\Phi_{x_2}(x_1^*, x_2^*) = 0 \quad \text{and} \quad x_1^* + Px_2^* = H.$$

Hence, $(x_1^*, x_2^*) \in \mathbb{R}^2_{++}$ is a solution to the system of equations (3). Uniqueness follows directly from the strict concavity of Φ. □

A.7 Proof of Example 2

It is straightforward to verify that $\Phi \in C^2(\mathbb{R}^3_{++})$ and that it satisfies assumptions (A_4), (A_5), (A_6), and (A_7). The system of equations in (6) can be written as

$$\begin{cases} \frac{1}{2}(x_1x_2)^{-\frac{1}{2}}x_2 + e^{-x_1} - \frac{1}{2}(x_1x_2)^{-\frac{1}{2}}x_1 - e^{-x_2} = 0 \\ P\left(\frac{1}{2}(x_1x_2)^{-\frac{1}{2}}x_2 + e^{-x_1}\right) - x_3^{-\frac{1}{2}} - e^{-x_3} = 0 \\ x_1 + x_2 + Px_3 = H \end{cases}$$

The first equation implies $x_1 = x_2$. By setting $u = x_1 = x_2$ and $x_3 = s$, we can write $u = \frac{H-Ps}{2}$ for $s \in]0, \frac{H}{P}[$. Then, the second equation becomes

$$P\left(\frac{1}{2} + e^{-u}\right) - \frac{1}{\sqrt{s}} - e^{-s} = 0.$$

We next define

$$\gamma(s) := P\left(\frac{1}{2} + e^{-u}\right) - \frac{1}{\sqrt{s}} - e^{-s}$$

and observe that $\gamma'(s) > 0$ for each $s \in]0, H/P[$, which implies that γ is strictly increasing. Furthermore, the following limits hold

$$\lim_{s\to 0} \gamma(s) = -\infty \quad \text{and} \quad \lim_{s\to \frac{H}{P}} \gamma(s) = \frac{3P}{2} - \sqrt{\frac{P}{H}} - e^{-\frac{H}{P}}.$$

Hence, the system of equations in (6) has a solution if and only if $\frac{3P}{2} - \sqrt{\frac{P}{H}} - e^{-\frac{H}{P}} > 0$. The result in the example follows straightforwardly. □

A.8 Proof of Theorems 4 and 5

We start by observing that, under additive separability, the system of equations (2) becomes

$$\begin{cases} p_i\, v_1'(x_1) - p_1\, v_i'(x_i) = 0, \text{ for each } i = 2, \ldots, n, \\ \mathbf{p} \cdot \mathbf{x} = H. \end{cases}$$

We want to show that this system has a unique solution $\mathbf{x} \in D$. The idea of the proof is to express each quantity x_i as a function of λ, the Lagrange multiplier associated with the budget constraint, and to apply the intermediate value theorem to the resulting expenditure function. Note that $\mathbf{x} \in D$ solves the system above if and only if there exists a $\lambda > 0$ such that $v_i'(x_i) = \lambda p_i$ for each $i = 1, \ldots, n$ and $\mathbf{p} \cdot \mathbf{x} = H$ (given a solution, it suffices to set $\lambda = v_1'(x_1)/p_1$, while the converse is immediate).

We first consider the bounded case. By the assumptions $v_i \in C^2(]a_i, b_i[)$, $v_i' > 0$, $v_i'' < 0$, and

$$\lim_{s\to a_i} v_i'(s) = +\infty, \qquad \lim_{s\to b_i} v_i'(s) = 0,$$

it follows that v_i' has a continuously differentiable, strictly decreasing inverse

$$w_i :]0, +\infty[\longrightarrow]a_i, b_i[$$

for each $i = 1, \ldots, n$. The same properties are inherited by the function

$$\lambda \in]0, +\infty[\;\longmapsto\; \sum_{i=1}^{n} p_i\, w_i(\lambda p_i),$$

which tends to $\mathbf{p} \cdot \mathbf{b}$ as $\lambda \to 0$ and to $\mathbf{p} \cdot \mathbf{a}$ as $\lambda \to +\infty$. Since this function is continuous and $\mathbf{p} \cdot \mathbf{a} < H < \mathbf{p} \cdot \mathbf{b}$, the intermediate value theorem ensures that there exists a $\lambda^* > 0$ such that $\sum_{i=1}^{n} p_i\, w_i(\lambda^* p_i) = H$, which is unique because

the function is strictly decreasing. Thus, the system of equations (2) has a unique solution, given by

$$x_i^* = w_i(\lambda^* p_i), \text{ for each } i = 1, \ldots, n.$$

This completes the proof of the bounded case. In the unbounded case, there is just one further issue to consider, since (A_7) is not assumed. Because v_i' is positive and strictly decreasing, the limit $\lim_{s\to+\infty} v_i'(s)$ exists and is non-negative. Denote it by ℓ_i. The inverse of v_i' is then $w_i :]\ell_i, +\infty[\to]0, +\infty[$, which is again continuous and strictly decreasing. Let ℓ be the largest of the values ℓ_i/p_i. The function above is then defined on $]\ell, +\infty[$, and it tends to $+\infty$ as $\lambda \to \ell$ and to 0 as $\lambda \to +\infty$. Since $H > 0$, the conclusion follows as in the bounded case. □

A.9 Viability of Amoroso's original proof strategy under separability

Let us introduce some additional notation. We shall denote by $\mathcal{F}^{as}(n)$ the class of functions Φ such that $\Phi(\mathbf{x}) = \sum_{i=1}^n v_i(x_i)$ and whose additive components v_i satisfy $v_i \in C^2(]a_i, b_i[)$ with $a_i < b_i$ and $0 < b_i < +\infty$, $v_i' > 0$, $v_i'' < 0$,

$$\lim_{s\to a_i} v_i'(s) = +\infty, \quad \text{and} \quad \lim_{s\to b_i} v_i'(s) = 0.$$

It is straightforward to note that, if $\Phi \in \mathcal{F}^{as}(n)$, then $\Phi \in \mathcal{F}(n)$. Furthermore, if $\Phi \in \mathcal{F}(n)$ and Φ is additively separable, then $\Phi \in \mathcal{F}^{as}(n)$.

Proof of Theorem 4. We closely follow Amoroso's proof strategy, employing induction on the number of commodities. Since Amoroso's proof overlooked some cases (Figures 2–4), for both the base for the induction and the inductive step, we deal with instances which are not in Amoroso (1928).

We assume, without loss of generality, that $p_1 = 1$, $a_i = 0$, and $b_i = 1$. For every $n \in \mathbb{N}$ and $\Phi \in \mathcal{F}^{as}(n+1)$, we aim to prove that, for any

$$\mathbf{p} = (1, p_2, \ldots, p_n, p_{n+1}) \in \mathbb{R}^{n+1}_{++}, \quad \text{and} \quad H \in]0,\, 1 + \sum_{i=2}^{n} p_i + p_{n+1}[,$$

the system of equations

$$\begin{cases} p_i\, \Phi_{x_1}(\mathbf{x}, x_{n+1}) - \Phi_{x_i}(\mathbf{x}, x_{n+1}) = 0, \text{ for each } i = 2, \ldots, n, \\ p_{n+1}\, \Phi_{x_1}(\mathbf{x}, x_{n+1}) - \Phi_{x_{n+1}}(\mathbf{x}, x_{n+1}) = 0 \\ x_1 + \sum_{i=2}^n x_i p_i = H - p_{n+1} x_{n+1} \end{cases} \tag{15}$$

has a solution in $(\mathbf{x}, x_{n+1}) \in]0, 1[^n \times]0, 1[$.

First we set up the base for the induction by solving (15) in dimension two. So we have

$$\Phi(x_1, x_2) = v_1(x_1) + v_2(x_2), \quad \mathbf{p} = (1, P), \quad 0 < H < 1 + P$$

where we set $p_2 = P$ for notational convenience. As before, there are four different cases to analyze which are depicted in Figures 1–4. Since Theorem 2 covers the case

in Figure 1, we concentrate on the three cases corresponding to Figures 2, 3, and 4. We start by considering the case of $P \leq H < 1$ (see Figure 3), where the domain of

$$g(s) := Pv_1'(H - Ps) - v_2'(s)$$

is $]0, 1[$. By assumption (A_6), it is immediate to verify that

$$\lim_{s \to 0} g(s) = Pv_1'(H) - \infty = -\infty.$$

By assumption (A_7), which is used here explicitly for the first time[24], it is immediate to verify that

$$\lim_{s \to 1} g(s) = \begin{cases} Pv_1'(H - P) - 0, \text{ if } H > P \\ \infty - 0, \text{ if } H = P \end{cases} > 0.$$

By applying the intermediate value theorem, we conclude that the AFT holds in this case. We next consider the case of $1 \leq H < P$ (see Figure 2). Since this is symmetric to the case of Figure 3, by the same argument used in the proof of Lemma 3, we conclude that the AFT holds also in this case. We finally consider the case $\max\{1, P\} \leq H < 1 + P$ (see Figure 4), where the domain of $g(s)$ is $]\frac{H-1}{P}, 1[\subset]0, 1[$. It is immediate to note that for $s \to 1$ the function $g(s)$ behaves as above. Then we just consider the left endpoint of the domain of g. So,

$$\lim_{s \to \frac{H-1}{P}} g(s) = 0 - \begin{cases} v_2'\left(\dfrac{H-1}{P}\right) \text{ if } H > 1 \\ \infty \text{ if } H = 1 \end{cases} < 0$$

Hence also in this case the AFT holds by applying the intermediate value theorem.

Now we have to prove the inductive step. Turning to the case of an arbitrary number of commodities, we set

$$\underline{\mathrm{H}} = \frac{H - (1 + \sum_{i=2}^{n} p_i)}{p_{n+1}} \quad \text{and} \quad \bar{H} = \frac{H}{p_{n+1}}.$$

Since $\underline{\mathrm{H}} < 1$, $\underline{\mathrm{H}} < \bar{H}$, and $0 < \bar{H}$, we shall work in the interval

$$]0, 1[\cap]\underline{\mathrm{H}}, \bar{H}[=]\max\{0, \underline{\mathrm{H}}\}, \min\{1, \bar{H}\}[.$$

This produces the following four interval cases:[25]

	$\underline{\mathrm{H}} < 0$	$\underline{\mathrm{H}} \geq 0$
$\bar{H} < 1$	$]0, \bar{H}[\quad (I_1)$	$]\underline{\mathrm{H}}, \bar{H}[\quad (I_2)$
$\bar{H} \geq 1$	$]0, 1[\quad (I_3)$	$]\underline{\mathrm{H}}, 1[\quad (I_4)$

[24] To be clear: we mean correctly used for the first time within the proof strategy of Amoroso, who used (A_7) in the erroneous inductive step; naturally, we have already used the assumption in the proof of Theorem 4 presented in Appendix A.8.

[25] Trivially, we observe that $\underline{\mathrm{H}} < 0 \iff H < 1 + \sum_{i=2}^{n} p_i$ and that $\bar{H} < 1 \iff H < p_{n+1}$.

The inductive hypothesis is as follows. Whenever $\Psi \in \mathcal{F}^{as}(n)$ and $\mathbf{q} = (1, q_2, \ldots, q_n) \in \mathbb{R}^n_{++}$, for every $K \in]0, 1 + \sum_{i=2}^n q_i[$ the system

$$\begin{cases} \mathbf{q} \cdot \mathbf{x} = K, \\ q_i \, \Psi_{x_1}(\mathbf{x}) - \Psi_{x_i}(\mathbf{x}) = 0, \text{ for each } i = 2, \ldots, n, \end{cases} \tag{16}$$

has a unique solution. Let us apply the inductive hypothesis by considering

$$\mathbf{q} = (1, p_2, \ldots, p_n) \quad \text{and} \quad \Psi(\mathbf{x}) = \Phi(\mathbf{x}, s), \text{ for each } s \in]0, 1[.$$

We then define the function $\mathbf{x}(K)$ such that $\mathbf{x} :]0, 1 + \sum_{i=2}^n p_i[\to]0, 1[^n$, which associates to any admissible K the solution to the following system of equations

$$\begin{cases} x_1 + \sum_{i=2}^n x_i p_i = K, \\ p_i \, v_1'(x_1) - v_i'(x_i) = 0, \text{ for each } i = 2, \ldots, n. \end{cases} \tag{17}$$

We observe that such a function $\mathbf{x}(K)$ is continuous by a careful application of the implicit function theorem.[26] Furthermore, we define the function

$$\eta :] \max\{\underline{\mathrm{H}}, 0\}, \min\{1, \bar{H}\}[\to]0, 1 + \sum_{i=2}^n p_i[$$

such that $\eta(s) = H - p_{n+1} s$ and the function $\mathbf{x}^*(s)$ which is given by the composition of $\mathbf{x}(K)$ and $\eta(s)$, *i.e.*, $\mathbf{x}^*(s) = \mathbf{x}(\eta(s))$. This means that for any admissible s, we have

$$x_1^*(s) + \sum_{i=2}^n x_i^*(s) p_i = H - p_{n+1} s,$$

and

$$p_i \, v_1' \left(x_1^*(s)\right) - v_i' \left(x_i^*(s)\right) = 0, \text{ for each } i = 2, \ldots, n.$$

To complete the proof we have to show that the continuous function

$$g(s) := p_{n+1} v_1' \left(x_1^*(s)\right) - v_{n+1}'(s),$$

has a zero in its domain $] \max\{\underline{\mathrm{H}}, 0\}, \min\{1, \bar{H}\}[$. To do so, we need to consider separately the four cases outlined above.

[26] Some comments are in order concerning the continuity of $\mathbf{x}(K)$. First, consider the system (16) only under the assumptions that $\Psi(\mathbf{x})$ is twice continuously differentiable, and that $\mathbf{q}$ has $q_1 \neq 0$, and that no other conditions are imposed. Under these assumptions, it is not difficult to prove that the Jacobian matrix with respect to the x_i-variables of the functions involved in the system is non-singular at every point where the Hessian of Ψ is negative definite. Note that when Ψ is, in particular, additively separable, the verification of this is quite straightforward and elementary. Finally, when $\Psi \in \mathcal{F}^{as}(n)$ the inductive hypothesis says that the system admits one and only one solution for every admissible $\mathbf{q}$ and K. Consequently, a careful application of the implicit function theorem guarantees that the solution globally depends continuously on the parameters.

Case I_1 The function g is defined on $]0, \bar{H}[\subset]0, 1[$, so we have to study the limits of g in 0 and $\bar{H} = \frac{H}{p_{n+1}} < 1$. It is immediate to see that $\lim_{s\to 0} \eta(s) = H$. Since in this case H belongs to the domain of $\mathbf{x}(K)$, it follows that $\lim_{s\to 0} x_1^*(s) = x_1(H)$. Consequently:

$$\lim_{s\to 0} g(s) = p_{n+1} v_1'(x_1(H)) - \infty = -\infty$$

By the budget equation, we have that $0 < x_1^*(s) < H - p_{n+1}s$. Then, it is immediate to conclude that $\lim_{s\to \bar{H}} x_1^*(s) = 0$. Consequently:

$$\lim_{s\to \bar{H}} g(s) = +\infty - v_{n+1}'\left(\bar{H}\right) = \infty$$

Case I_2 The function g is defined on $]\underline{\mathrm{H}}, \bar{H}[\subset]0, 1[$ and with respect to Case I_1 the only change is that we have to investigate the behavior of g close to $\underline{\mathrm{H}} = \frac{H - \left(1 + \sum_{i=2}^n p_i\right)}{p_{n+1}} \geq 0$. Again, by the budget equation, we have:

$$H - \sum_{i=2}^n p_i - p_{n+1}s < H - \sum_{i=2}^n x_i^*(s) p_i - p_{n+1}s = x_1^*(s) < 1$$

Then, it is immediate to conclude that $\lim_{s\to \underline{\mathrm{H}}} x_1^*(s) = 1$. Consequently:

$$\lim_{s\to \underline{\mathrm{H}}} g(s) = 0 - \begin{cases} v_{n+1}'(\underline{\mathrm{H}}), \text{ if } \underline{\mathrm{H}} > 0 \\ \infty, \text{ if } \underline{\mathrm{H}} = 0 \end{cases} < 0.$$

Case I_3 The function g is defined on $]0, 1[$ so with respect to Case I_1 the change is that we have to investigate the behavior of g close to 1. Since $\bar{H} \geq 1$, it follows that $H \geq p_{n+1}$. When $H = p_{n+1}$, by the budget equation, we have that $0 < x_1^*(s) < H(1 - s)$. Then, it is immediate to conclude that $\lim_{s\to 1} x_1^*(s) = 0$. When $H > p_{n+1}$, we have that $0 < H - p_{n+1} < 1 + \sum_{i=2}^n p_i$, so $H - p_{n+1}$ belongs to the domain of $\mathbf{x}(K)$. Consequently:

$$\lim_{s\to 1} g(s) = \begin{cases} +\infty - 0, \text{ if } H = p_{n+1} \\ p_{n+1} v_1'(x_1(H - p_{n+1})) - 0, \text{ if } H > p_{n+1} \end{cases} > 0.$$

Case I_4 The function g is defined on $]\underline{\mathrm{H}}, 1[\subset]0, 1[$. So the behavior of g at the endpoints of the domain has been described already in Case I_2 and Case I_3.

Hence, the intermediate value theorem can be applied in all cases and then the system of equations in (15) has a solution. Uniqueness follows directly from the strict concavity of Φ. Therefore, the statement of the theorem easily follows. □

Proof of Theorem 5. Proving Theorem 5 amounts to considering the case in which $a_i = 0$ and $b_i = +\infty$ for each $i = 1, \ldots, n$. The proof is a simplified version of the previous one, since there are no separate cases to consider, either in the base or in the inductive step. With a slight abuse of notation, we use the same symbol $\mathcal{F}^{as}(n)$ for the class of functions Φ such that $\Phi(\mathbf{x}) = \sum_{i=1}^n v_i(x_i)$ with $v_i \in C^2(]0, +\infty[)$, $v_i' > 0$, $v_i'' < 0$, and $\lim_{s\to 0} v_i'(s) = +\infty$.

We assume again, without loss of generality, that $p_1 = 1$. For every $n \in \mathbb{N}$ and $\Phi \in \mathcal{F}^{as}(n+1)$, we aim to prove that, for any

$$\mathbf{p} = (1, p_2, \ldots, p_n, p_{n+1}) \in \mathbb{R}^{n+1}_{++}, \quad \text{and} \quad H > 0,$$

the system of equations (15) has a solution in $(\mathbf{x}, x_{n+1}) \in]0, \infty[^n \times]0, \infty[$.

Since the base for the induction is already covered by the result in Theorem 3, we move directly to the inductive step. The inductive hypothesis is as follows. Whenever $\Psi \in \mathcal{F}^{as}(n)$ and $\mathbf{q} = (1, q_2, \ldots, q_n) \in \mathbb{R}^n_{++}$, for every $K > 0$ the system of equations (16) has a unique solution. As before, we apply the inductive hypothesis by considering

$$\mathbf{q} = (1, p_2, \ldots, p_n) \quad \text{and} \quad \Psi(\mathbf{x}) = \Phi(\mathbf{x}, s), \text{ for each } s \in]0, +\infty[,$$

and we consider the continuous function $\mathbf{x}(K)$ such that $\mathbf{x} :]0, \infty[\to]0, \infty[^n$, which associates to any admissible K the solution to the system of equations (17). Furthermore, we consider the function $\eta :]0, \bar{H}[\to]0, \infty[$ such that $\eta(s) = H - p_{n+1}s$ and the function $\mathbf{x}^*(s)$ which is given by the composition of $\mathbf{x}(K)$ and $\eta(s)$, *i.e.*, $\mathbf{x}^*(s) = \mathbf{x}(\eta(s))$. This means that for any admissible s, we have

$$x_1^*(s) + \sum_{i=2}^{n} x_i^*(s) p_i = H - p_{n+1}s,$$

and

$$p_i \, v_1' \left(x_1^*(s)\right) - v_i' \left(x_i^*(s)\right) = 0, \text{ for each } i = 2, \ldots, n.$$

To complete the proof we have to show that the continuous function

$$g(s) = p_{n+1} v_1' \left(x_1^*(s)\right) - v_{n+1}'(s),$$

has a zero in the domain $]0, \bar{H}[$. This can be proved analogously to the above Case I_1 and, as above, uniqueness follows directly from the strict concavity of Φ. Therefore, the statement of the theorem easily follows. □

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